Sharpness in the Fuzzy World

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RAMiCS 2020, October 28, 2020

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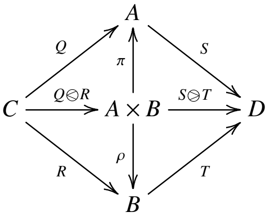
- Motivation.
- Categories of L-Fuzzy Relations.
- Fuzzy Sharpness Theorem.

Motivation I

Sharpness problem of relational products

$$Q \olessthan R := Q;\pi^{\smile} \sqcap R;\rho^{\smile} \quad S \ogreaterthan T := \pi;S \sqcap \rho;T$$

$$(Q \olessthan R);(S \ogreaterthan T) = Q;S \sqcap R;T$$



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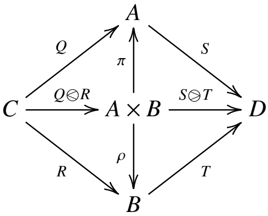

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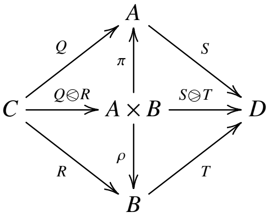

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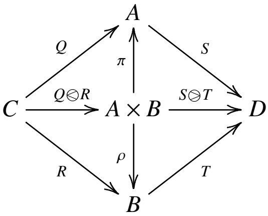

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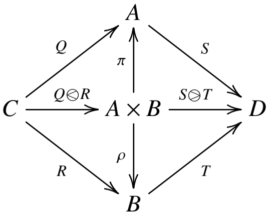

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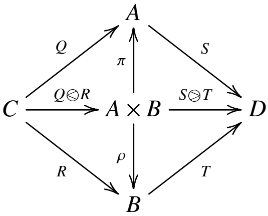

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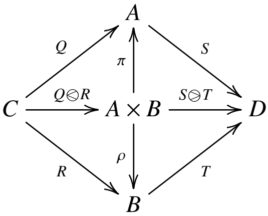

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  - additional products exist (H. Zierer and J. Desharnais),
  - one relation is of type $B \to \mathcal{P}(B)$ (G. Schmidt and M. Winter).

Motivation II

Sharpness problem in the Fuzzy World

$Q \olessthan_* R := Q;\pi^\smile * R;\rho^\smile \quad S \ogreaterthan_* T := \pi;S * \rho;T$

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For concrete relations and a t-norm like operation $*$:

$$(U * V)(x,y) := U(x,y) * V(x,y)$$

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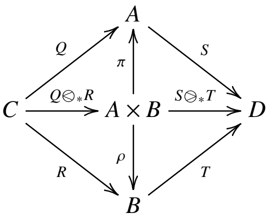

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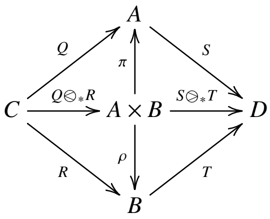

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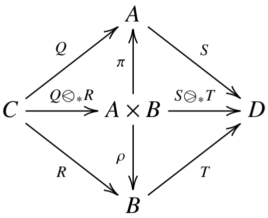

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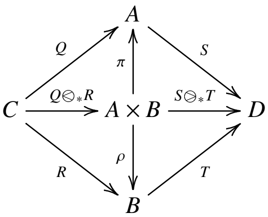

- Again easy to verify for concrete relations.
- Trivial: Not true in all relational catgeories (Reglar sharpness is special case).
- Additional challenge: Less properties available since $*$ is not idempotent.

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- Generalize J. Desharnais's theorem to the fuzzy case.
  - The proof uses the modular inclusion in its full generality multiples times. Only weaker versions of the modular inclusion for $\overset{*}{;}$ and $*$ are valid.
- Generalize G. Schmidt's and M. Winter's theorem to the fuzzy case.

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- Generalize J. Desharnais's theorem to the fuzzy case.
  - The proof uses the modular inclusion in its full generality multiples times. Only weaker versions of the modular inclusion for $\overset{*}{;}$ and $*$ are valid.
- Generalize G. Schmidt's and M. Winter's theorem to the fuzzy case.
  - We will show that this is possible.
  - This version of the theorem will then be used to reduce the fuzzy sharpness problem to a regular one.

Dedekind Categories

Definition

A Dedekind category $\mathcal{R}$ is a category satisfying the following:

1. For all objects A and B the collection $\mathcal{R}[A,B]$ is a complete distributive lattice with operations $\sqcap, \sqcup, \sqsubseteq, \perp\!\!\!\perp_{AB}, \top\!\!\!\top_{AB}$.
2. There is a monotone operation $\smile$ (called conversion) so that for all relations $Q : A \to B$ and $R : B \to C$:
$$(Q;R)^{\smile} = R^{\smile};Q^{\smile}, \quad (Q^{\smile})^{\smile} = Q.$$
3. For all relations $Q : A \to B, R : B \to C$ and $S : A \to C$ the modular law holds:
$$Q;R \sqcap S \sqsubseteq Q;(R \sqcap Q^{\smile};S).$$
4. For all relations $R : B \to C$ and $S : A \to C$ there is a relation $S/R : A \to B$ (called the left residual of S and R) so that for all $Q : A \to B$ the following holds:
$$Q;R \sqsubseteq S \iff Q \sqsubseteq S/R.$$

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ideal relations — scalar relations

Definition

1. A relation $J : A \to B$ is called an ideal iff $\pi_{AA}; J; \pi_{BB} = J$.
2. A relation $\alpha : A \to A$ is called a scalar on A iff $\alpha \sqsubseteq \mathbb{I}_A$ and $\pi_{AA}; \alpha = \alpha; \pi_{AA}$.

Arrow Categories

Definition

An arrow category $\mathcal{A}$ is a Dedekind category with $\mathbb{T}_{AB} \neq \perp\!\!\!\perp_{AB}$ for all objects A and B together with two operations $^{\uparrow}$ and $^{\downarrow}$ satisfying the following:

1. $R^{\uparrow}, R^{\downarrow} : A \to B$ for all $R : A \to B$.
2. $(^{\uparrow}, ^{\downarrow})$ is a Galois correspondence.
3. $(R^{\smile}; S^{\downarrow})^{\uparrow} = R^{\uparrow\smile}; S^{\downarrow}$ for all $R : B \to A$ and $S : B \to C$.
4. $(Q \sqcap R^{\downarrow})^{\uparrow} = Q^{\uparrow} \sqcap R^{\downarrow}$ for all $Q, R : A \to B$.
5. If $\alpha_A \neq \perp\!\!\!\perp_{AA}$ is a non-zero scalar, then $\alpha_A^{\uparrow} = \mathbb{I}_A$.

Fuzzy Categories

Definition

A fuzzy category $\mathcal{F}$ is an arrow category together with two operations $*$ and $\mathbin{;\!\!*}$, so that the following holds:

1. $*$ maps two relations $Q : A \to B$ and $R : A \to B$ to a relation $Q * R : A \to B$.
2. $*$ is associative, commutative and continuous.
3. $Q * R^{\downarrow} = Q \sqcap R^{\downarrow}$ for all $Q, R : A \to B$.
4. $(Q * R)^{\smile} = Q^{\smile} * R^{\smile}$ for all $Q, R : A \to B$.
5. $\mathbin{;\!\!*}$ maps two relations $Q : A \to B$ and $R : B \to C$ to a relation $Q \mathbin{;\!\!*} R : A \to C$.
6. $\mathbin{;\!\!*}$ is associative and continuous.
7. $Q \mathbin{;\!\!*} R^{\downarrow} = Q; R^{\downarrow}$ for all $Q : A \to B$ and $R : B \to C$.
8. $(Q \mathbin{;\!\!*} R)^{\smile} = R^{\smile} \mathbin{;\!\!*} Q^{\smile}$ for all $Q : A \to B$ and $R : B \to C$.

Fuzzy Categories

Definition

9. The exchange inclusion $(Q * R) \fatsemi (S * T) \sqsubseteq Q \fatsemi S * R \fatsemi T$ is valid for all $Q, R : A \to B$ and $S, T : B \to C$.
10. The following versions of the modular inclusion are valid:
    1. $Q; R * S \sqsubseteq Q; (R \sqcap Q^{\smile} \fatsemi S)$,
    2. $Q \fatsemi R * S \sqsubseteq Q; (R * Q^{\smile} \fatsemi S)$,
    3. $(P * Q) \fatsemi R * S \sqsubseteq P \fatsemi (R * Q^{\smile} \fatsemi S)$,

    for all $P, Q : A \to B, R : B \to C$, and $S : A \to C$.

Relational Products

Definition

An object $A \times B$ together with relations $\pi : A \times B \to A$ and $\rho : A \times B \to B$ is called a relational product of A and B iff

$$\pi, \rho \text{ are crisp}, \quad \pi^{\smile};\pi \sqsubseteq \mathbb{I}_A, \quad \rho^{\smile};\rho \sqsubseteq \mathbb{I}_B, \quad \pi^{\smile};\rho = \mathbb{T}_{AB}, \quad \pi;\pi^{\smile} \sqcap \rho;\rho^{\smile} = \mathbb{I}_{A\times B}.$$

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$$\pi = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad \rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Relational Powers

Definition

An object $\mathcal{P}(A)$ together with a relation $\varepsilon : A \to \mathcal{P}(A)$ is called a relational power iff

$$\mathrm{syQ}(\varepsilon, \varepsilon)^{\downarrow} = \mathbb{I}_{\mathcal{P}(A)} \text{ and } \mathrm{syQ}(R, \varepsilon)^{\downarrow} \text{ is total for every } R : A \to B.$$

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$$\varepsilon = \begin{pmatrix} 0 & 0 & 0 & a & a & a & 1 & 1 & 1 \\ 0 & a & 1 & 0 & a & 1 & 0 & a & 1 \end{pmatrix}$$

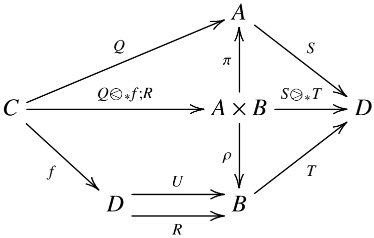
Fuzzy Sharpness for Concrete Relations

Theorem

If $Q : C \to A, R : C \to B, S : A \to D$ and $T : B \to D$ are L-relations and $\pi : A \times B \to A$, $\rho : A \times B \to B$ the standard (concrete) projection relations, then we have:

$$(Q \otimes_* R);_*(S \otimes_* T) = Q;_*S * R;_*T.$$

Fuzzy Sharpness Theorem I

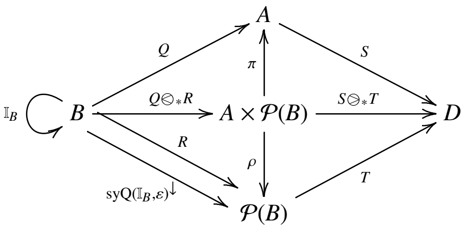


Theorem

Let $\mathcal{F}$ be a fuzzy category, $A \times B$ the relational product of A and B, $Q : C \to A, R : D \to B, S : A \to E, T : B \to E$ be relations, and $f : C \to D$ be crisp and univalent. If there is an injective relation $U : D \to B$ such that $f;U$ is total, then we have:

$$(Q \olessthan_* f;R) \mathbin{\overset{*}{;}} (S \ogreaterthan_* T) = Q \mathbin{\overset{*}{;}} S * f;(R \mathbin{\overset{*}{;}} T).$$

Fuzzy Sharpness Theorem II



Corollary

Let $\mathcal{F}$ be a fuzzy category, $\mathcal{P}(B)$ the relational power of B, $A \times \mathcal{P}(B)$ the relational product of A and $\mathcal{P}(B)$, and $Q : B \to A, R : B \to \mathcal{P}(B), S : A \to D, T : \mathcal{P}(B) \to D$ be relations. Then we have

$$(Q \olessthan_* R) ; (S \ogreaterthan T) = Q ; S * R ; T.$$

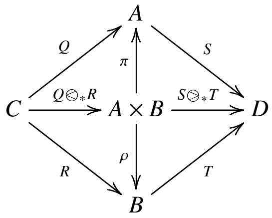
Fuzzy Sharpness Theorem III

Please note that the previous corollary immediately implies

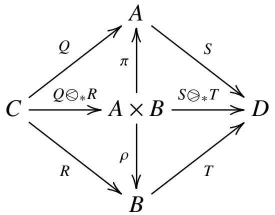
$$(\varepsilon \olessthan_* \varepsilon) \stackrel{*}{;} (S \ogreaterthan_* T) = \varepsilon \stackrel{*}{;} S * \varepsilon \stackrel{*}{;} T$$

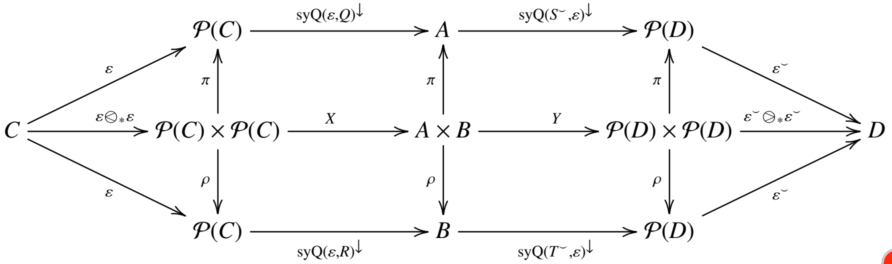
for appropriate relations S and T.

Fuzzy Sharpness Theorem IV



Fuzzy Sharpness Theorem IV





Fuzzy Sharpness Theorem V

Theorem

Let $\mathcal{F}$ be a fuzzy category, and be $A \times B$ the relational product of A and B. If the objects $\mathcal{P}(C), \mathcal{P}(D), \mathcal{P}(C) \times \mathcal{P}(C), \mathcal{P}(D) \times \mathcal{P}(D)$ exist, and if the product $A \times B$ is sharp for all crisp relations $Q' : \mathcal{P}(C) \times \mathcal{P}(C) \to A$, $R' : \mathcal{P}(C) \times \mathcal{P}(C) \to B$, $S' : A \to \mathcal{P}(D) \times \mathcal{P}(D)$, and $T' : B \to \mathcal{P}(D) \times \mathcal{P}(D)$, i.e., we have

$$(Q' \otimes R');(S' \otimes T') = Q';S' \sqcap R';T',$$

then also the fuzzy version of sharpness holds, i.e., we have

$$(Q \otimes_* R) \mathbin{;_*} (S \otimes T) = Q \mathbin{;_*} S * R \mathbin{;_*} T.$$

for all $Q : C \to A, R : C \to B, S : A \to D$, and $T : B \to D$.

Fuzzy Sharpness Theorem VI

Corollary

Let $\mathcal{F}$ be a fuzzy category with relational products and powers. Then fuzzy sharpness holds.

Thank you
for your attention!

Questions?