Stone Dualities from Opfibrations

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Background: Stone dualities

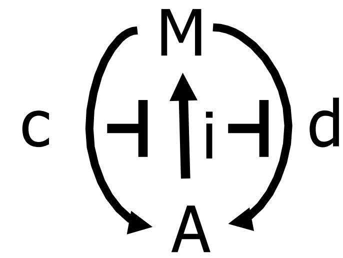
- dual equivalences between certain categories of algebras and those of topological spaces
- derived from fundamental adjunctions by cutting down unnecessary objects
- Examples
  - $\mathbf{Sober}^{op} \cong \mathbf{SpatialFrm}$ ... derived from $\mathrm{Point} \dashv \mathrm{Open} : \mathbf{Top}^{op} \to \mathbf{Frm}$
  - $\mathbf{Stone}^{op} \cong \mathbf{BoolAlg}$ ... derived from $\mathrm{PrimeFilter} \dashv \mathrm{Clop} : \mathbf{Top}^{op} \to \mathbf{BoolAlg}$ (left adj. :f.f.)

Our aim and main result

- Our aim:
  - to construct fundamental adjunctions without concrete topological arguments
- Input data of our main construction
  - A,M,B: categories of algebras, subalgebras, parameters
  - five functors among them satisfying certain conditions
- Output data
  - the category FS of formal spaces
  - the fundamental adjunction between A and FS^{op}

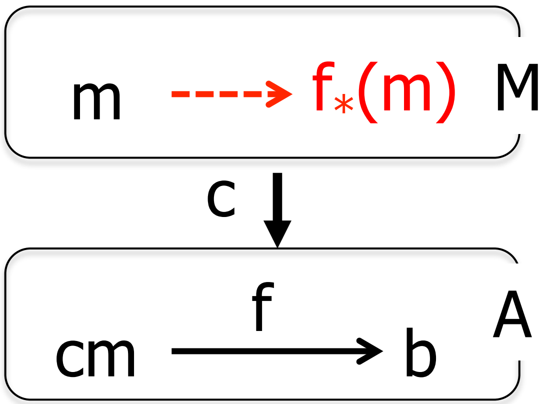
Opfibred comprehension

- (Def.) An opfibred comprehension is (c:M→A, i, d) s.t.
  - c:M→A is an opfibration
  - i: A→M is the right adjoint to c whose counit is identity
  - d: M→A is the right adjoint to i whose unit is identity



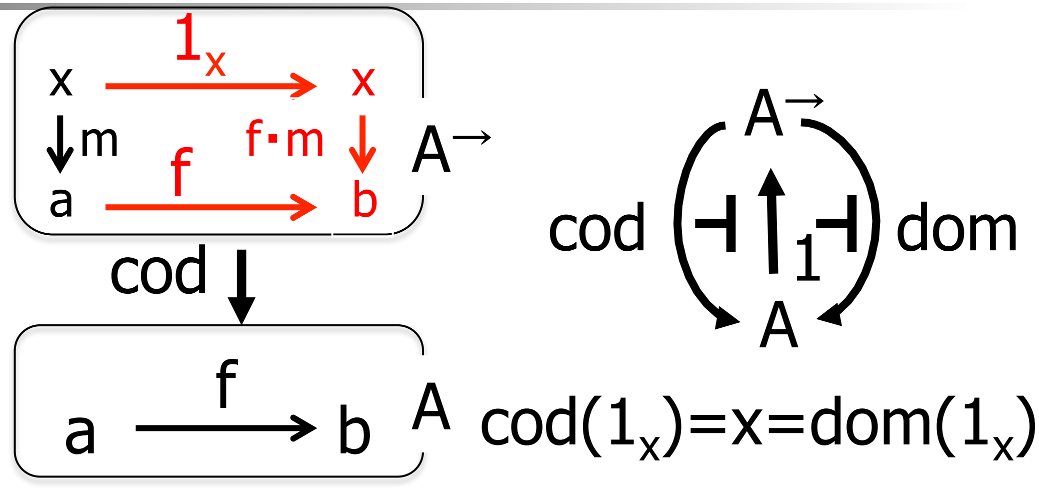
$c(i(x))=x=d(i(x))$

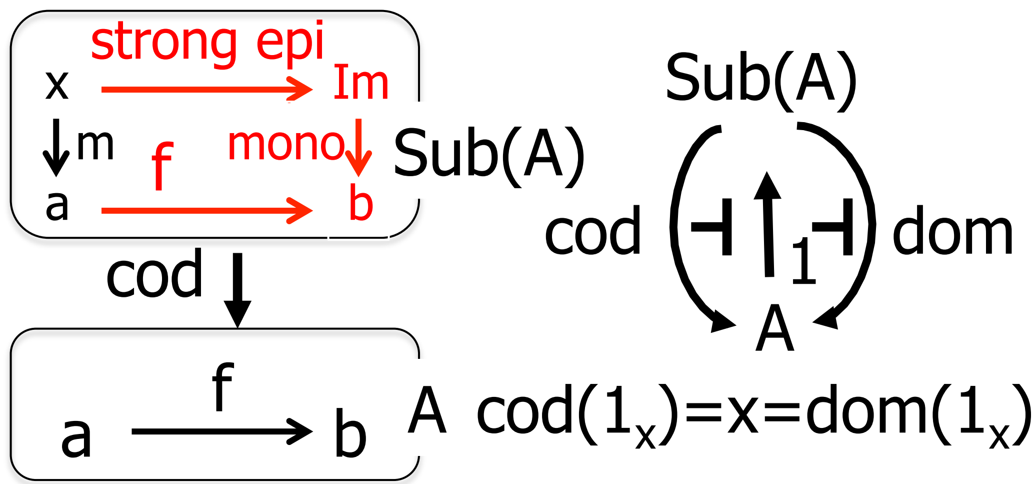
- (Def.) An opfibration is a functor satisfying the cocartesian lifting property.



Opfibred comprehension

- Examples
  1. $M = A^{\rightarrow}$
     : the arrow category
  2. $M = Sub(A)$
     : only mono
     when A has
     (strong epi, mono)
     -factorization
  3. (E,M): factorization
     system on A



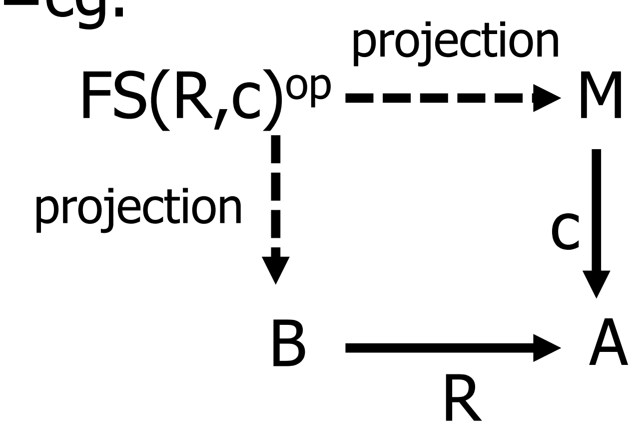


Formal spaces

(c:M→A, i, d) : opfibred compehension

$L \dashv R : B \to A$: adjunction

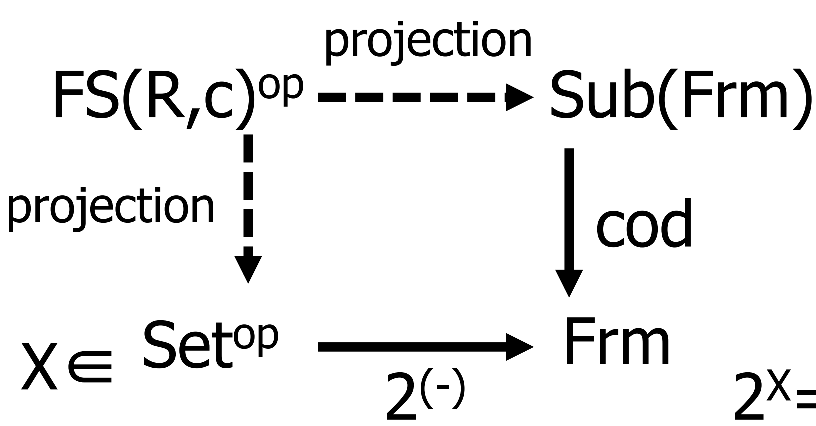
- (Def.) FS(R,c) is the category of the followings.
  - A formal space is (b∈B, m∈M) s.t. Rb=cm.
  - A formally continuous map $(b_1,m_1) \to (b_2,m_2)$ is $(f:b_2 \to b_1,\ g:m_2 \to m_1)$ s.t. Rf=cg.



- $FS(R,c)^{op}$ is the pullback of R and c in Cat.

The leading example: topological spaces:

- $FS(2^{(-)},cod)=Top$ (topological spaces and cont. maps)
  - Frm : the category of frames
    (frame : complete join lattice with distributive finite meets)
  - $(cod:Sub(Frm)\to Frm,1,dom)$: opfibred comprehension
  - $Frm(-,2) \dashv 2^{(-)} : Set^{op}\to Frm$ where 2^X is the powerset of X



$\ni$ m: subframe of 2^X

$\Leftrightarrow m\subseteq 2^X$ closed for $\cup$ and finite $\cap$

$\Leftrightarrow$ m is the set of open sets on X

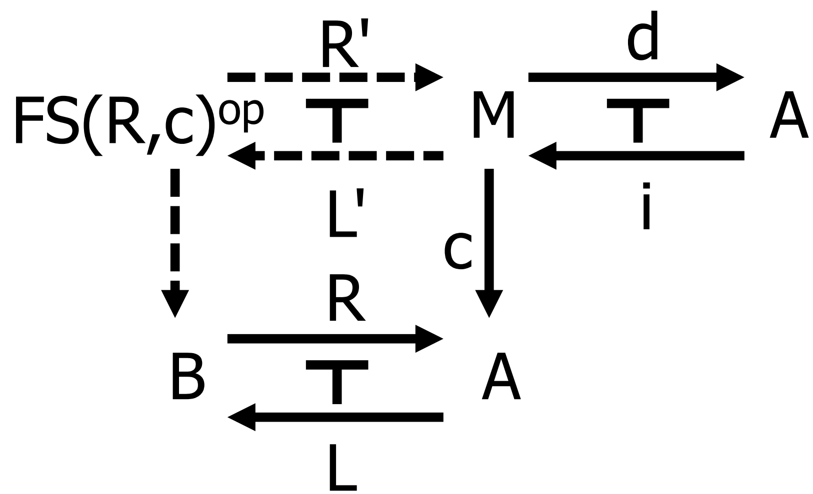
$2^X=cod(m)$

Other examples of FS(R,c)

- Fld
  - fields of sets (subalgebras of powerset Boolean algebras)
  - c=cod : Sub(BoolAlg)→BoolAlg, R=$2^{(-)}$: Set^{op} →BoolAlg
- PoTop
  - topological spaces with partial order
    (open sets must be up-closed)
  - c=cod : Sub(Frm)→Frm, R=Up : $Poset^{op}$ →Frm
- Chu
  - Chu spaces
  - c=cod : $A^{\rightarrow}$ →A, R=[-,Ω] : A^{op} →A

Fundamental adjunctions

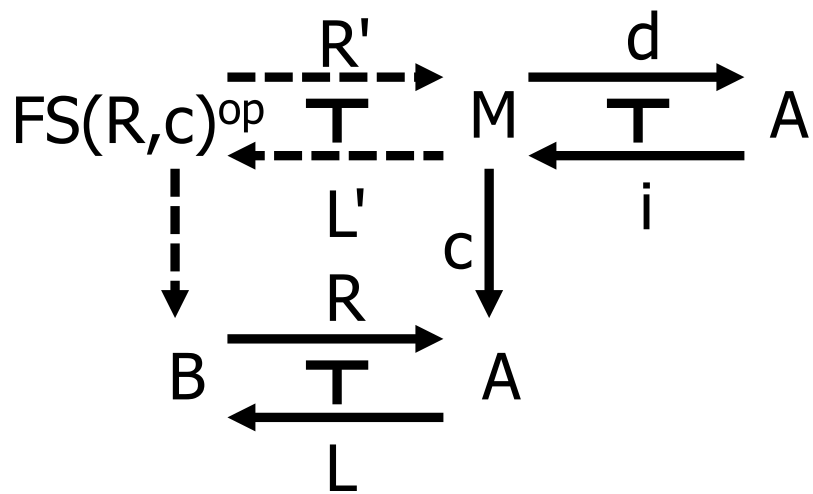
- (Construction) $L'\cdot i \dashv d\cdot R'$: fundamental adjunction.
  - $FS(R,c)^{op}$ is a pullback of R and c
  - An opfibration lifts adjunctions onto pullbacks[Hermida93]



- Example
  - Point $\dashv$ Open : $Top^{op} \rightarrow Frm$

When the left fundamental adjoint is fully faithful

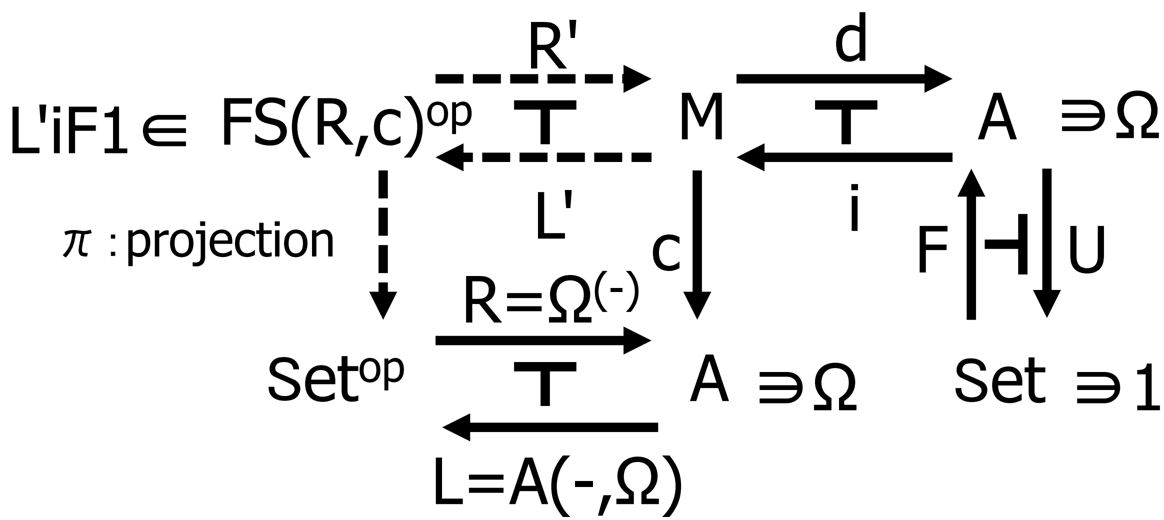
- (Prop.) The unit of $L \dashv R$ belongs to M
  ⇔ The left fundamental adjoint $L' \cdot i$ is fully faithful.



- Examples
  - Any arrow in A belongs to $M=A^{\to}$.
  - An arrow in A belongs to M=Sub(A) ⇔ it is mono.
    (e.g. units of $\mathrm{BoolAlg}(-,2) \dashv 2^{(-)}$, $\mathrm{DLat}(-,2) \dashv 2^{(-)}$)

When the fundamental adjoints are representable

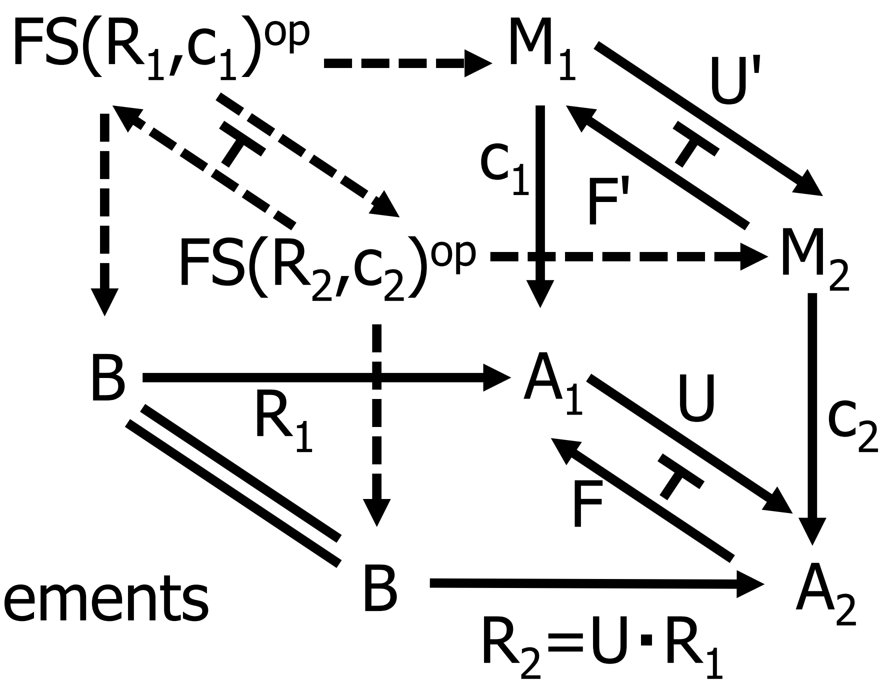
- (Prop.) If A has F ⊣ U : A→Set and A(-,Ω)=L ⊣ R=Ω$^{(-)}$: Set^{op}→A for some Ω∈A, then the fundamental adjoints are representable.
  - π・L'・i = A(-,Ω) : A^{op}→Set
  - U・d・R' ≅ FS(R,c)(-,L'iF1) : FS(R,c)^{op}→Set



To get adjunctions between different formal spaces

(Construction) adjunction between $FS(R_1,c_1)$ and $FS(R_2,c_2)$ from

- $(c_1,i_1,d_1),(c_2,i_2,d_2)$: opfibred compehension
- (c_1,c_2) : map of adjunction from $F' \dashv U'$ to $F \dashv U$
- $R_2=U \cdot R_1$



- Example:
  A_1=DLat, A_2=BoolAlg,
  M_1=Sub(A_1),M_2=Sub(A_2),
  F : forgetful functor,
  U returns complemented elements

Conclusion and Future Work

- Conclusion
  - construction of fundamental adjunctions
    without concrete topological arguments
  - input data: opfibred comprehension on A and adjunction
  - output data: adjunction between A and its formal spaces
- Future Work
  - Priestley duality
  - relationship with coalgebras
  - natural duality [Clark, Davey 1998]

Thank you