The involutive (or Girard) quantaloid of completely distributive lattices

aka

Cyclic and dualizing sup-preserving endomaps of a complete lattice[1]

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[1] Including results from [LS, ACT2020]

Plan

Linear negations, in nature

Raney's transform, categorically

Cyclic and dualizing elements of $[L, L]_\vee$

Conclusions, future works

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Quantales of sup-preserving endofunctions

For L in Sup-Latt, the homset

$$[L, L]_\vee$$

has a canonical structure of a quantale.

[LS, RAMICS 2018] If L is a finite chain or $L = [0, 1]$, then $[L, L]_\vee$ is an involutive quantale (or Girard quantale).

That is,

$$O_L(x) := \bigvee_{t<x} t$$

is a cyclic and dualizing element.

Problem *For which L, complete chain and/or lattice, is $[L, L]_\vee$ an involutive quantale?*

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Recaps on Girard quantales

- **Quantale**: monoid (or semigroup) in Sup-Latt,
- **Girard quantale**: star-autonomous, possibly non-symmetric but cyclic, complete poset.

A possible axiomatization:

$$f = f^{**}\,,$$
$$f \circ g \leq h \quad \text{iff} \quad f \leq h/g \quad \text{iff} \quad g \leq f\backslash h\,,$$

with

$$h/g := (g \circ h^*)^*\,, \qquad f\backslash h := (h^* \circ f)^*\,.$$

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You should listen to this talk if ...

- You are an enthusiastic linear logician ...
- ... and/or love weakening relations:

$$[\mathcal{D}(P), \mathcal{D}(P)]_\vee \simeq \text{quantale of weakening relations on } P.$$

- You wonder about join-endomorphisms.
- The category of relations is not big enough for you:

$$\mathsf{Rel} \subseteq \mathsf{Sup\text{-}Latt}\,.$$

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When $[L, L]_\vee$ is a Girard quantale?

For L a complete lattice, define the **Raney's transform** as

$$g^\vee(x) := \bigvee_{x \not\leq t} g(t)\,, \quad \text{for each } g : L \to L.$$

Define then

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Proposition. *If L is a* **completely distributive lattice**, *then $[L, L]_\vee$ is, with this structure, a Girard quantale.*

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Girard quantales II, and a theorem by Eklund et al.

Definition Let Q be a quantale. $0 \in Q$ is said to be

- **dualizing** if $f = 0/(f\backslash 0) = (0/f)\backslash 0$, for each $f \in Q$,
- **cyclic** if $0/f = f\backslash 0$, for each $f \in Q$.

Lemma *Let $f^* := f\backslash 0$. Then $(Q, ^*)$ is a Girard quantale iff 0 is cyclic and dualizing.*

Theorem [Eklund et al., 2018] *Let*

$$O_L := {id_L}^{\vee} \in [L, L]_{\vee} .$$

Then O_L is dualizing iff L is a completely distributive lattice.

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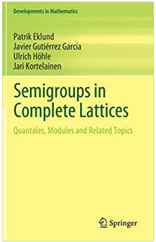

Plan

Sup-Latt as a star-autonomous category

Recall that Sup-Latt is star-autonomous with

$$X^* := [X, \mathbf{2}]_\vee \simeq X^{op}\,.$$

Let

$$[X, Y]_\wedge := \{\, f : X \to Y \mid f \text{ is inf-preserving} \,\}$$

with the pointwise ordering.

Then

$$[X, Y]_\wedge = [X^{op}, Y^{op}]_\vee \simeq [X^*, Y^*]_\vee^* \simeq^1 Y \otimes X^*\,.$$

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[1] Recall that $\neg(X \Rightarrow Y) = \neg Y \wedge X$

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$$[X, Y]_\wedge \xrightarrow{\ \vee\ } [X, Y]_\vee$$

is the map

$$Y \otimes X^* \xrightarrow{\ \mathrm{mix}_{Y,X^*}\ } Y \oplus X^*$$

transpose of

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See [Higgs and Rowe, 1989, Cockett and Seely, 1997].

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Bits of Raney's theorem(s) (1960)

Theorem [Raney, 1960] *The map*

$$\mathrm{mix}_{L,L^{op}} : L \otimes L^{op} \to [L, L]_\vee$$

is surjective if and only if L is completely distributive.

Since mix is natural, the image of mix is a bi-ideal. Consequently:

Proposition *TFAE :*

1. *the map* $\mathrm{mix}_{L,L^{op}}$ *is surjective,*
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Cyclic elements of $[L, L]_\vee$

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- $[L, L]_\vee$ *has exactly two central elements,* $\bot$ *and* id_L*.*
- $[L, L]_\vee$ *has at most two cyclic elements,* $\top$ *and* ${id_L}^\vee$*.*
- *If* ${id_L}^\vee$ *is cyclic, then L is completely distributive (and so* ${id_L}^\vee$ *is dualizing as well)*

Further remarks

Theorem [LS, RAMICS 2020]

- $[L, L]_\vee \models 0 \leq 1$ *if and only if* L *is a chain.*
- $[L, L]_\vee \models 1 \leq 0$ *if and only if* L *has no completely join-prime elements.*
- *If an involutive residuated lattice embeds into a Girard quantale of the form* $[L, L]_\vee$, *then it is distributive.*
- *The full subcategory of Sup-Latt whose objects are the completely distributive lattices is a Girard quantaloid (in a unique way).*

Remark [Rowe, 1988, Higgs and Rowe, 1989] The full subcategory of Sup-Latt whose objects are the completely distributive lattices is compact closed.

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A useful statement (folklore?)

Proposition *In any Girard quantale (or involutive residuated lattice), the mapping $f \mapsto f^*$ bijectively sends*

- *cyclic elements to central elements,*
- **dualizing elements to invertible elements**,

see [LS, ACT2020] *but also* [Kelly and Laplaza, 1980].

Dualizing elements of $[L, L]_\vee$

Proposition [LS, ACT2020] *If $f \in [L, L]_\vee$ is dualizing, then $\rho(f)^\vee$ is invertible and, consequently, L is a completely distributive lattice.*

Thus, if $[L, L]_\vee$ is a Frobenius quantale (if it has a dualizing element) then $id_L{}^\vee$ is cyclic and dualizing and $[L, L]_\vee$ is a Girard quantale.

Theorem [LS, ACT2020] *$f \in [L, L]_\vee$ is dualizing* **iff** *L is completely distributive and $\rho(f)^\vee$ is invertible.*

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Next steps: when $[L, L]_\vee$ is autodual ?

Since

$$[X, Y]_\vee^{op} \simeq^\rho [Y^{op}, X^{op}]_\vee^{op} \simeq X \otimes Y^{op} = [Y, X]_\wedge \,,$$

maps of the form

$$[X, Y]_\vee^{op} \longrightarrow [X, Y]_\vee$$

bijectively correspond, via precomposition with ρ, to maps of the form

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This is exactly what happens in the formula

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$$[X, Y]_\vee^{op} \longrightarrow [X, Y]_\vee$$

bijectively correspond, via precomposition with ρ, to maps of the form

$$[Y, X]_\wedge \longrightarrow [X, Y]_\vee \,.$$

This is exactly what happens in the formula

$$f^* = [\,\rho(f)\,]^\vee \,.$$

Finite lattices

Theorem [LS, ACT2020] *If L is finite and $[L, L]_\vee$ is autodual, then L is distributive.*

Problem If $[L, L]_\vee$ is autodual, is L necessarily a completely distributive lattice?

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Plan

Linear negations, in nature

Raney's transform, categorically

Cyclic and dualizing elements of $[L, L]_\vee$

Conclusions, future works

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- Cyclic and dualizing elements of $[L, L]_\vee$ completely characterized.
- Girard quantale structure on $[L, L]_\vee$ strongly canonical (when L is completely distributive):
  1. unique,
  2. extends to a Girard quantaloid,
  3. implied from Frobenius quantale structure.

- Get back to equational theory of $[C, C]_\vee$, when C is a chain.
- Compare the notion of Girard quantaloid with that of a compact closed category.
- Axiomatize the category Sup-Latt and its full subcategory of completely distributive lattices.
- Show that $[L, L]_\vee$ autodual implies L completely distributive.
- Use knowledge of $[L, L]_\vee$ and its structure for counting.

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Thanks for your attention !!!

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