Weakening relation algebras and FL^2-algebras

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Outline

- Classical relation algebras
- Full Lambek algebras (FL-algebras)
- Cyclic involutive FL-algebras
- FL^2-algebras
- Generalized bunched implications algebras (GBI)
- Weakening relations (wRA and RwRA)
- Congruences on FL^2-algebras

Classical algebras of binary relations

The **calculus of binary relations** was developed by

A. De Morgan [1864], **C. S. Peirce** [1883], and **E. Schröder** [1895]

It was / is considered one of the **cornerstones** of mathematical logic

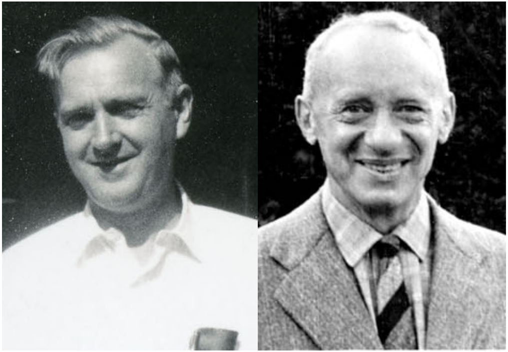
B. Russell [1903]:
"The subject of symbolic logic is formed by three parts: the calculus of propositions, the calculus of classes, and the **calculus of relations**."

Alfred Tarski [1941] gave a set of axioms, refined in 1943 to **10 equational axioms**, for (abstract) **relation algebras**

Classical algebras of binary relations

Jónsson-Tarski [1948 Bulletin of the AMS, Abstract 89]:

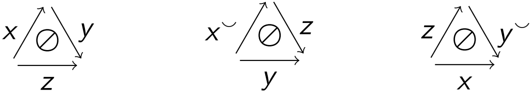
A **relation algebra** (RA) A is a **Boolean algebra** with a binary **associative operator** ; and a unary **operator** $^\smile$ such that ; has the **unit element** 1, $x^{\smile\smile} = x$, $(xy)^\smile = y^\smile x^\smile$ and $x^\smile ; \neg(x;y) \leq \neg y$



Relation algebras, residuated lattices, FL-algebras

$\mathbf{A} = (A, \wedge, \vee, \bot, \top, \neg, ;, 1, ^\smile)$ is a **relation algebra (RA)** if $(A, \wedge, \vee, \bot, \top, \neg)$ is a **Boolean algebra**, $(A, ;, 1)$ is a **monoid** and

$$x;y \leq \neg z \iff x^\smile; z \leq \neg y \iff z; y^\smile \leq \neg x$$



equivalently $\quad x;y \leq z \iff y \leq \neg(x^\smile; \neg z) \iff x \leq \neg(\neg z; y^\smile)$

$\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /)$ is a **residuated lattice (RL)** if $(A, \wedge, \vee)$ is a **lattice**, $(A, \cdot, 1)$ is a **monoid** and $\backslash, /$ are the **left** and **right** **residuals** of $\cdot$, i.e.,

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y$$

A **FL-algebra** is a residuated lattice with an additional **constant** 0 (denoting any element).

Involutive FL-algebras

The 0 is used to define the **linear negations** $\sim x = x\backslash 0$ and $-x = 0/x$.

An **involutive FL-algebra (InFL)** is an FL-algebra s.t. $\sim - x = x = - \sim x$

It follows that $x\backslash y = \sim(-y \cdot x)$ and $x/y = -(y \cdot \sim x)$.

A **CyInFL-algebra** is an InFL-algebra that is **cyclic**: $\sim x = -x$

E.g. a **relation algebra** $(A, \wedge, \vee, \neg, \cdot, ^{\smile}, 1)$ is a **CyInFL-algebra** if one defines $x\backslash y = \neg(x^{\smile} \cdot \neg y)$, $x/y = \neg(\neg x \cdot y^{\smile})$ and $0 = \neg 1$, and omits the operations $\neg, ^{\smile}$ from the signature

The **cyclic linear negation** is given by $\sim x = \neg(x^{\smile}) = (\neg x)^{\smile}$

The variety **CyInFL** is **noncommutative MALL** and has a **decidable equational theory** while this is **not the case** for relation algebras

Generalizing relation algebras

Note: $\mathbf{A} \in \mathbf{RA} \implies (A, \wedge, \vee, \top, \neg) \in \mathbf{BA}$ and $(A, \wedge, \vee, ;, 1, \sim) \in \mathbf{CyInFL}$

BA is the **logical part**, and **CyInFL** is the **dynamic part**.

We now replace **BA** and/or **CyInFL** with larger varieties.

In the **most general setting**, we have $\mathbf{FL}^2$**-algebras** where the logical **and** the dynamic part are FL-algebras.

Bunched implication (BI-)algebras have a **Heyting algebra** as logical part and a **commutative residuated lattice** as dynamic part.

BI-logic is the propositional part of **separation logic**, a **Hoare-logic** for modeling memory heaps, mutable data structures and concurrent processes.

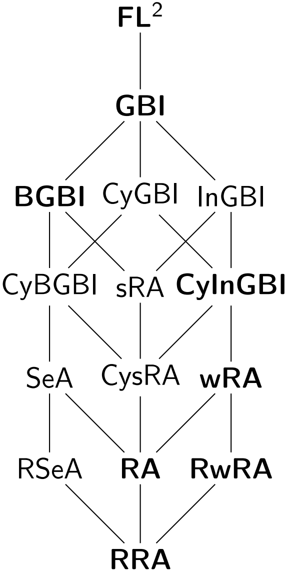
Overview of varieties

$\mathsf{RL} = \mathsf{Lat} + \mathsf{Mon} + \text{Residuation}$
$\mathsf{FL} = \mathsf{RL} + {\sim}x = x\backslash 0,\ -x = 0/x$
$\mathsf{HA} = \mathsf{FL} + xy = x \wedge y,\ \neg x = x \to \bot$

In: ${\sim}{-x} = x = -{\sim}x$
Cy: ${\sim}x = -x$
$\mathsf{BA} = \mathsf{HA} + \neg\neg x = x$

Variety	Logical part	Dynamic part
FL^2-algebras	FL	FL
Gen. bunched implication algebras (GBI)	HA	RL
Cyclic involutive GBI-algebras (CyInGBI)	HA	CyInFL
Boolean GBI-algebras (BGBI)	BA	RL
Relation algebras (RA)	BA	CyInFL$+(xy)^{\smile} = y^{\smile}x^{\smile}$

Some inclusions among varieties



Number of nonisomorphic algebras

Number of elements: $n =$	2	3	4	5	6	7	8
Residuated lattices (RL)	**1**	**3**	20	149	1488	18554	295292
Gen. BI-algebras (GBI)	**1**	**3**	20	115	899	7782	80468
Cyclic involutive FL-algebras	**1**	**2**	9	21	101	279	1433
Cyclic inv. GBI-algebras (CyGBI)	**1**	**2**	9	8	43	48	281
Boolean gen. BI-algebras (BGBI)	**1**	**0**	**5**	0	0	0	83
Relation algebras (RA)	**1**	**0**	**3**	0	0	0	13

$n = 2$: (Generalized) Boolean algebra $\mathbf{2}$

$n = 3$: Heyting algebra $\mathbf{3}$, MV-algebra $\mathbf{Ł}_3$, Sugihara algebra $\mathbf{S}_3$

$n = 4$: $\mathbf{2} \times \mathbf{2}$, $\mathcal{P}(\mathbb{Z}_2)$, $\mathcal{S}(\mathbb{Z}_3)$, $\mathcal{P}(\mathbb{M}_2)$, $\mathcal{P}(\mathbb{P}_2)$

$\mathbb{Z}_n =$ 1-gen group, $\mathbb{M}_n =$ 1-gen monoid, $\mathbb{P}_n =$ maximally partial monoid

$\mathcal{S}(G) = \{S \subseteq G \mid x \in S \Rightarrow -x \in S\}$ for an abelian group G

Representable relation algebras

A relation algebra **A** is **representable**, or in **RRA**, if **A** is isomorphic to a subalgebra of $(\mathcal{P}(E), \cap, \cup, \emptyset, E, {}^c, ;, id_X, {}^\smile)$ for some set X and $E \subseteq X^2$.

Here $\quad R^c = E \setminus R \quad$ and $\quad R^\smile = \{(x, y) \mid (y, x) \in R\}$.

RRA is a variety [Tarski '56]

Let the **full RA on** X be $Rel(X) = (\mathcal{P}(X^2), \cap, \cup, \emptyset, X^2, {}^c, ;, id_X, {}^\smile)$.

Then $\mathbf{RRA} = \mathbf{HSP}\{Rel(X) : X \text{ is countable}\}$.

RRA is nonfinitely axiomatizable [Monk '64]

For more details see [Givant 2017] and [Maddux 2006]

Weakening relation algebras

For a poset $\mathbf{P} = (P, \leq)$, let $\mathrm{Wk}(\mathbf{P}) = \{R \subseteq P^2 \mid {\leq};R;{\leq} \subseteq R\}$.

Relations in $\mathrm{Wk}(\mathbf{P})$ are called **weakening closed relations** since

$$x \leq u \; R \; v \leq y \implies x \; R \; y$$

$\sim R := (R^c)^\smile = \{(y,x) \mid (x,y) \notin R\}$, the **complement-converse** of R.

Weakening relations are closed under **complement-converse**, **union**, **intersection**, Heyting **implication** $\to$ (= residual of intersection), relation **composition** ; and **residuals** $\backslash, /$ of composition.

$1 := \leq$ is a weakening relation and is the **identity** of composition.

The **full weakening relation algebra** on a poset $\mathbf{P}$ is

$$\mathbf{Wk}(\mathbf{P}) = (\mathrm{Wk}(\mathbf{P}), \cap, \cup, \to, P^2, \emptyset, ;, \sim, 1, 0), \text{ where } 0 = \sim 1.$$

Representable weakening RAs = **RwRA** = **HSP**$\{\mathbf{Wk}(\mathbf{P}) \mid \mathbf{P} \text{ is a poset}\}$.

A small example

Let $\mathbf{C}_2 = \{0, 1\}$ be the two element chain with $0 < 1$.

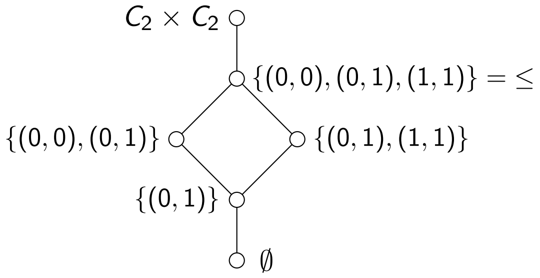


Figure: The full weakening relation algebra $\mathbf{Wk}(\mathbf{C}_2)$

A slightly bigger example

For the 3-element chain $\mathbf{C}_3 = \{0, 1, 2\}$, $\mathbf{Wk}(\mathbf{C}_3)$ has 20 elements

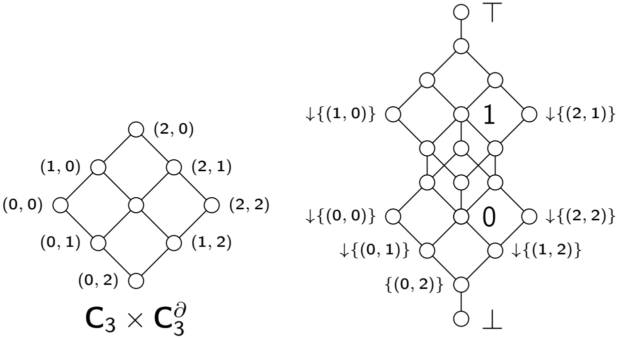


Figure: The full weakening relation algebra $\mathbf{Wk}(\mathbf{C}_3)$

Double division conuclei

An **interior operator** δ on a poset is an order-preserving map such that $\delta(\delta(x)) = \delta(x) \leq x$.

An interior operator δ is a **conucleus** if $\delta(x)\delta(y) \leq \delta(xy)$.

The conucleus **image** $\delta(\mathbf{A})$ of a residuated lattice is a residuated lattice $(\delta(A), \wedge_\delta, \vee, \cdot, \backslash_\delta, /_\delta)$ without 1, where $x *_\delta y = \delta(x * y)$ for $* \in \{\wedge, \backslash, /\}$.

Let $p \in A$ be a **positive idempotent**, i.e., $p = p^2 \geq 1$.

Then $\delta_p(x) = p\backslash x/p$ is a conucleus called the **double division conucleus**.

Lemma

$\delta_p(\mathbf{A}) = \{pxp \mid x \in A\}$, *and* p *is the identity element.*

Double division conuclei of relation algebras

In a full relation algebra, a positive idempotent p is a **preorder** $\mathbf{P} = (P, \sqsubseteq)$ (i.e., $p = \sqsubseteq$ is reflexive and transitive).

If $p \wedge p^{\smile} = 1$ then $\mathbf{P}$ is a poset and $\mathbf{Wk}(\mathbf{P}) = \delta_p(\mathrm{Rel}(P))$.

Hence the variety **RwRA** of representable weakening relation algebras contains all double division conucleus images of members of **RRA**.

For a class $\mathcal{K}$ of algebras let $\mathrm{d}\mathcal{K} = \{\delta_p(\mathbf{A}) : \mathbf{A} \in \mathcal{K}, 1 \leq p^2 = p \in A\}$.

Theorem (Galatos, J. 2020a)

If $\mathcal{V}$ is a variety of bounded residuated lattices or bounded GBI-algebras with $\top \backslash x / \top$ as unary discriminator on the subdirectly irreducible members then $S(\mathrm{d}\mathcal{V})$ is a discriminator variety with the same unary discriminator term.

Applying this result to the variety **RA** produces the discriminator variety **wRA** $:= S(\mathrm{d}\mathbf{RA})$ that contains both **RA** and **RwRA**.

Some identities that hold in **wRA**

Recall that the variety **RA** of **relation algebras** is an abstract counterpart of the variety **RRA** of **representable relation algebras**.

The variety **wRA** $= S(\mathrm{d}\mathbf{RA})$ generated by double-division conucleus images of relation algebras is the abstract counterpart of **RwRA**.

Problem: Find a (finite?) axiomatization of **wRA**.

In a GBI-algebra let the **domain** $d(x) = x\top \wedge 1$ and **range** $r(x) = \top x \wedge 1$.

Theorem

The identities

$d(x)x = x, \quad xr(x) = x, \quad \top x \top x \top = \top x \top$ *and* $\sim\neg(xy) \leq (\sim\neg y)(\sim\neg x)$

hold in ***wRA****.*

FL^2-algebras and bounded GBI-algebras

A **$\mathbf{FL}^2$-algebra** is of the form $\mathbf{A} = (A, \wedge, \vee, \diamond, \to, \leftarrow, t, f, \cdot, \backslash, /, 1, 0)$ s. t.

$$\mathbf{A}_t = (A, \wedge, \vee, \diamond, \to, \leftarrow, t, f) \text{ and } \mathbf{A}_1 = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$$

are FL-algebras.

Relation algebras are examples of **classical** FL^2-algebras: $\mathbf{A}_t$ is a **Boolean algebra** with $x \wedge y = x \diamond y$.

A **bounded GBI-algebra** is a FL^2-algebra that satisfies $x \wedge y = x \diamond y$, $t = \top$, $f = \bot$ and $0 = 1$.

A **bunched implication algebra**, or **BI-algebra**, is a commutative bounded GBI-algebra (i.e., $xy = yx$).

Congruences of residuated lattices

A **congruence filter** of a residuated lattice **A** is a subset of the form $F = \uparrow([1]_\theta)$ where θ is a congruence.

Congruence filters satisfy the following **normality condition** for $a \in A$ (where quantifiers range over F):

$$\forall x \in F\ \exists x_1, x_2 \in F,\ x_1 a \le ax \ \text{ and } \ ax_2 \le xa. \qquad (N_a)$$

A filter F satisfies (N) if (N_a) holds for all $a \in A$.

The set of **congruence filters** of **A** is denoted by CF(**A**).

Theorem (Blount, Tsinakis 2003)

For a residuated lattice **A***, a subset* F *is a congruence-filter if and only if* F *is a lattice filter and a submonoid of* **A** *that satisfies* (N)*.*
Moreover, Con(**A**) *is isomorphic to the lattice* CF(**A**) *of congruence-filters via the bijection* $\theta \mapsto \uparrow([1]_\theta)$ *and* $F \mapsto \{(x, y) : x/y, y/x \in F\}$*.*

Congruences of FL^2-algebras

For FL^2 the congruence 1-filters are determined by a stronger t-**normality** condition. For any $a \in A$

$$\forall x \in F,\ \exists x_1, x_2, x_3, x_4 \in F, \qquad (N_a^t)$$
$$ax_1 \leq a \diamond xt, \quad x_2 a \leq xt \diamond a, \quad a \diamond x_3 t \leq xa, \quad x_4 t \diamond a \leq ax$$

A filter F satisfies (N^t) if (N_a^t) holds for all $a \in A$.

Theorem (Galatos, J. 2020)

For an FL^2-algebra $\mathbf{A}$*, a subset* F *is the* 1*-filter of some congruence* θ *of* $\mathbf{A}$ *if and only if* F *is a lattice filter and* $\cdot, 1$*-submonoid of* $\mathbf{A}$ *that satisfies* (N^t)

An analogous result holds for congruence t-filters $\uparrow([t]_\theta)$ of FL^2-algebras.

Congruences of GBI-algebras

The previous result specializes to GBI-algebras:

Corollary

The 1-filters of a GBI-algebra **A** *are the filter submonoids that are closed under the terms*

$$u_a(x) = a\backslash(a \wedge x\top),\ v_a(x) = (a \to xa)/\top \textit{ and } \rho_a(x) = ax/a,$$

A previously known characterization of the congruence classes of GBI-algebras used more complicated terms with two parameters.

Similar 1-parameter terms exist for congruence $\top$-filters of GBI-algebra.

Theorem (Galatos, J. 2020a)

For an involutive GBI-algebra, a lattice filter F is a $\top$-filter if and only if for all $x \in F$ it follows that $\neg{\sim}x, \quad \neg{-}x, \quad {\sim}(\top(-x)\top) \in F$.

Some applications and conclusion

Weakening relations are the **analogue of binary relations** when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions.

Since sets can be considered as **discrete posets** (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a **substantial generalization** of binary relations.

They have applications in **sequent calculi**, **proximity lattices/spaces**, **order-enriched categories**, **cartesian bicategories** (**profunctors**), **bi-intuitionistic modal logic**, **mathematical morphology** and **program semantics**, e.g. via **separation logic**.

[Galatos, J. 2020a] also investigate a **meet-conucleus** (relativization) and a **multiplication-conucleous**.

There are many opportunities for interesting research projects.

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Thanks!