Preorders, partial semigroups, and quantales

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RAMiCS

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Background

- Algebraic structures applied to program verification and shortest path problems
  - Kleene algebra $\leftarrow$ Iterative operation $*$ is included
  - unital quantale
- Def.
  A unital quantale is a tuple $(Q, \leq, \vee, \odot, 1)$ s.t.
  - $(Q, \leq, \vee)$ is a complete join semilattice
  - $(Q, \odot, 1)$ is a monoid
  - for each element a and each subset S of Q
    - $a \odot (\vee S) = \vee\{a \odot b \mid b \in S\}$
    - $(\vee S) \odot a = \vee\{b \odot a \mid b \in S\}$

Since the Kleene algebra $*$ operation can be constructed from the join and the monoid structure, it is useful for modeling similar structures.

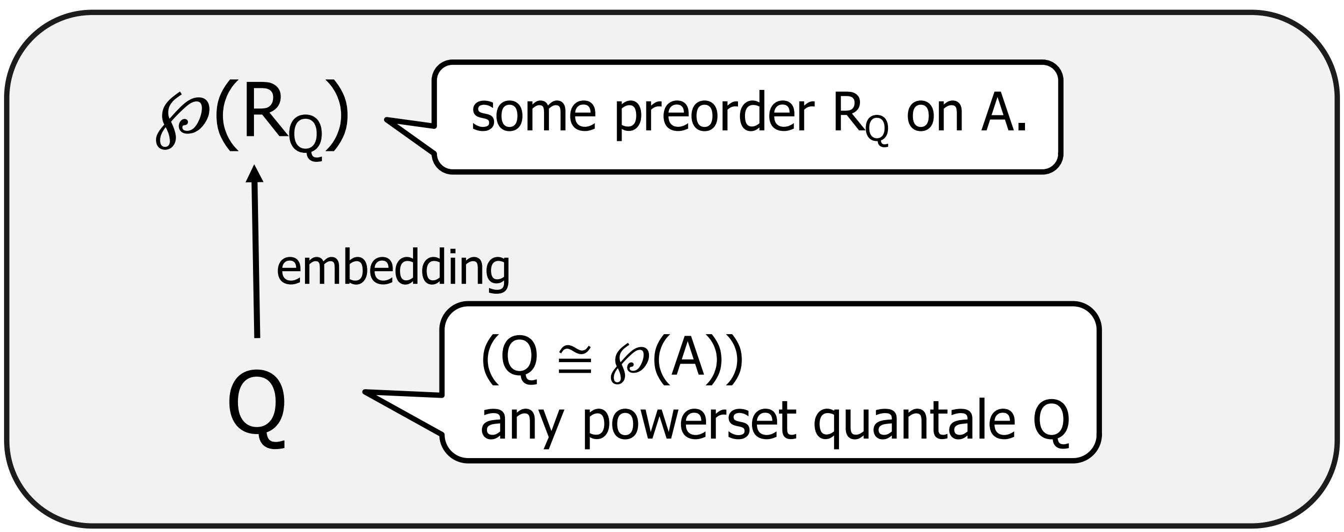
Background

- E.g. powerset quantale
  - A Powerset quantale is a unital quantale whose complete join semilattice is a powerset of some set.
  - Examples
    - $(\wp(A),\subseteq,\cup,\cap,A)$ ··· powerset
    - $(\wp(\leq),\subseteq,\cup,\circ,\Delta_A)$ ··· relational quantale
      - $\leq$: Preorder on A(which is seen as a set of pairs)
      - $Q \circ R=\{(a,c) \mid \exists b.\ (a,b)\in Q,\ (b,c)\in R\}$ (composition of binary relations)
      - $\Delta_A=\{(a,a) \mid a \in A\}$(identity relation)
    - $(\wp(A^*),\subseteq,\cup,\circledcirc,\{\varepsilon\})$ ··· The whole language

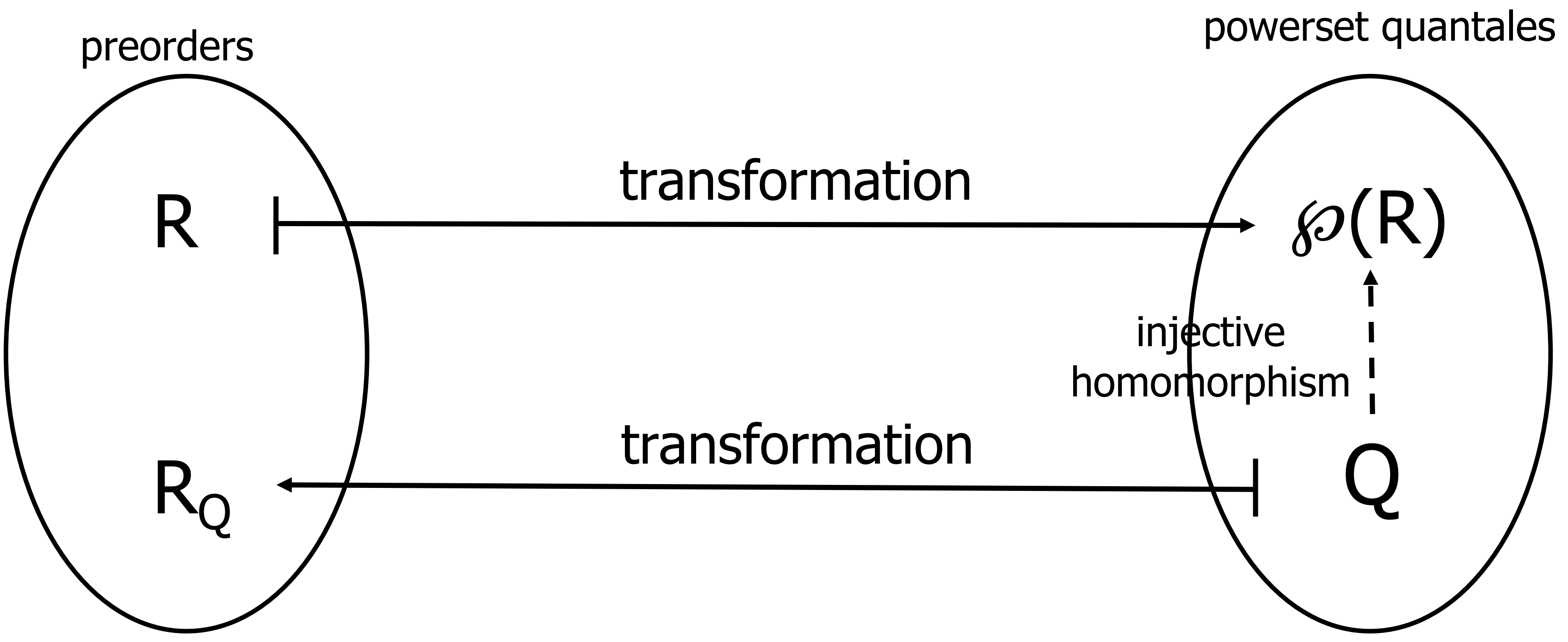
A relational quantale can embed any powerset quantale, and can be treated as a typical example.

Background

- Relational representation theorem
  - What kind of quantale can be embedded in the relational quantale by what kind of map.
  - Any powerset quantale can be embedded by an injective homomorphism. [Nishizawa・Furusawa 2012]

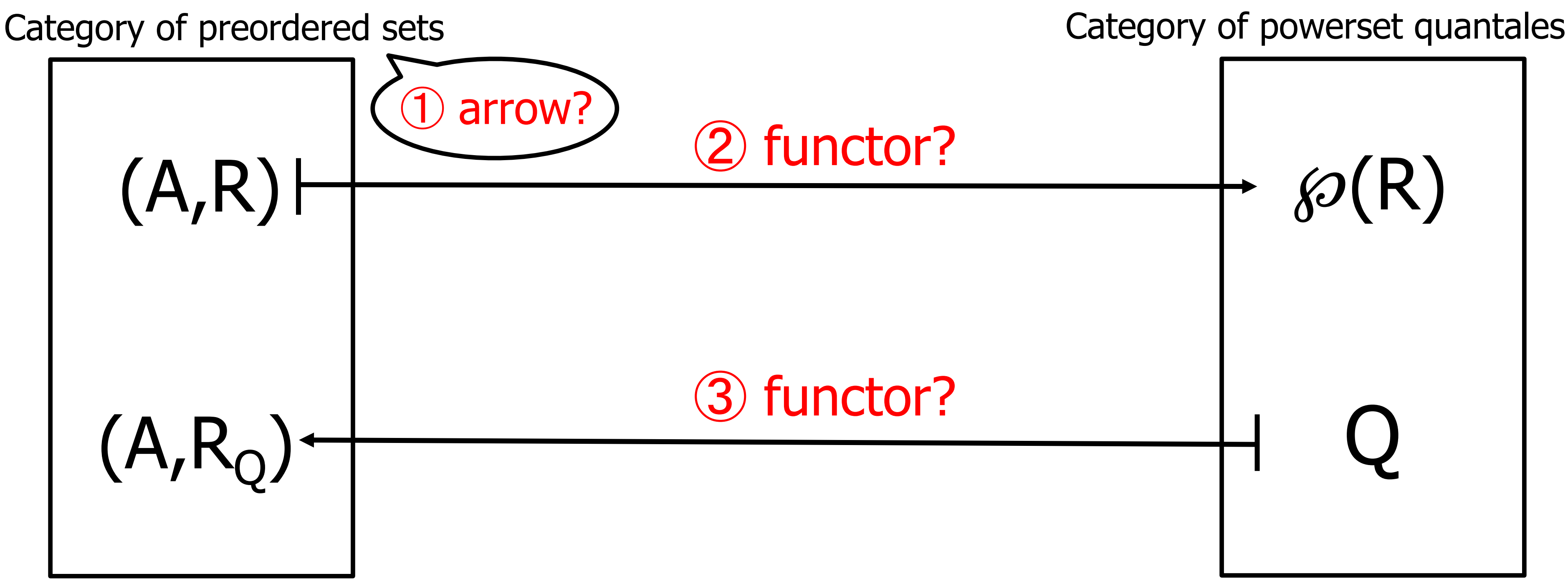


Background


preorders
powerset quantales
transformation
R
℘(R)
injective
homomorphism
transformation
R_Q
Q

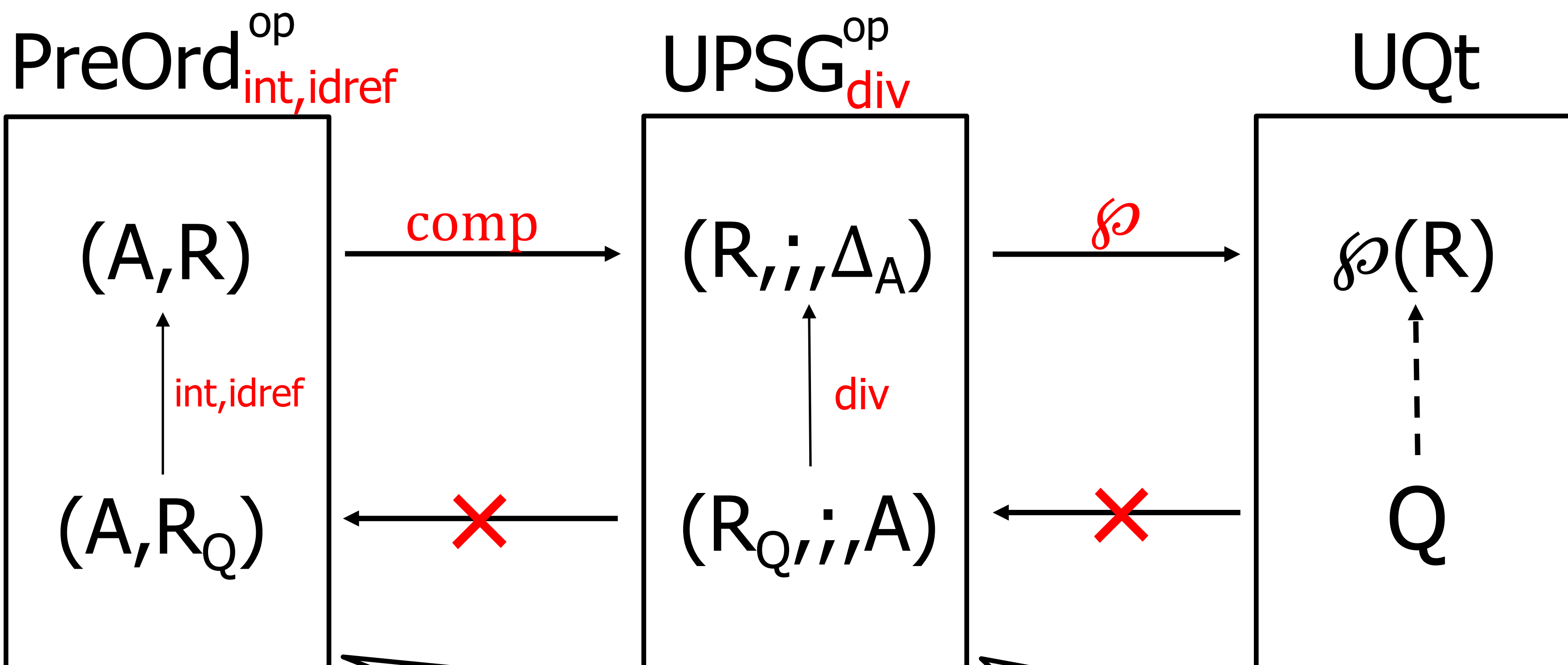
Goal

- Define them as categories and functors



- Finding ①, ②, ③

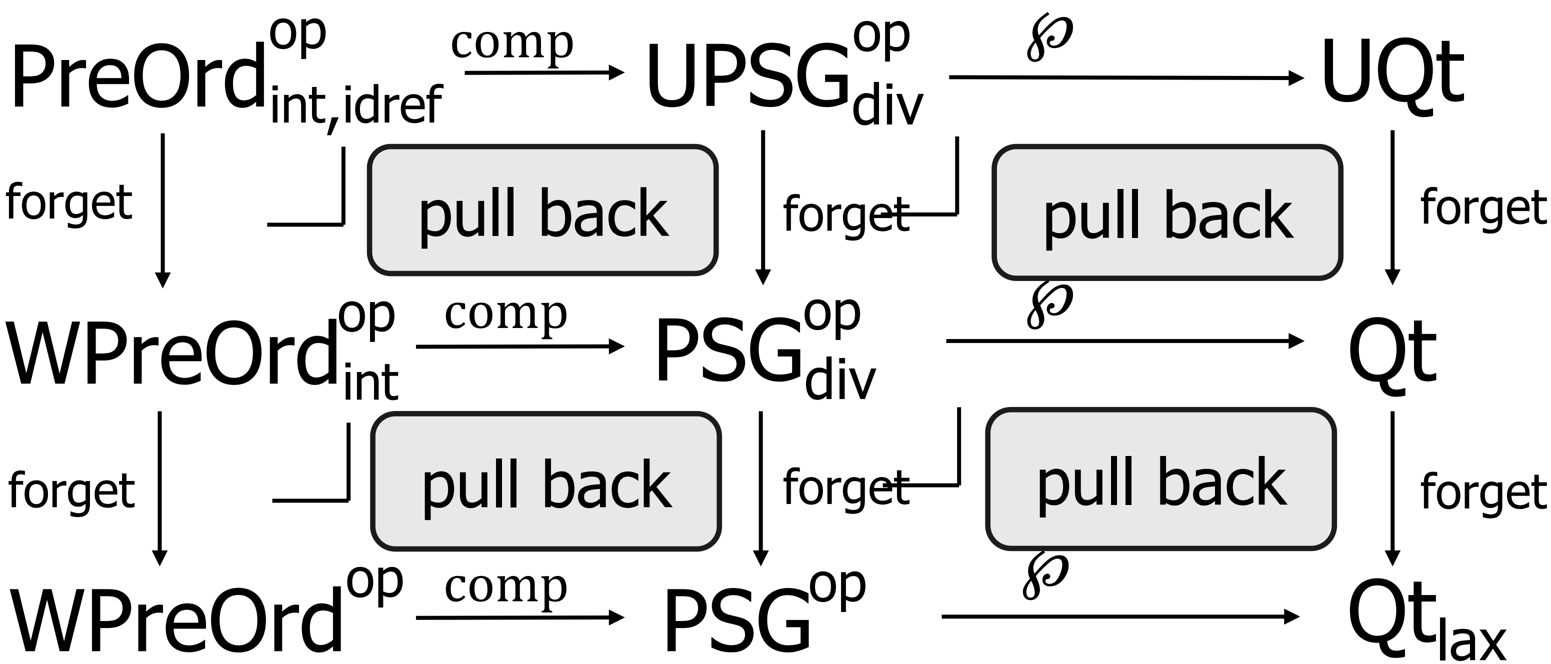
This paper's Results



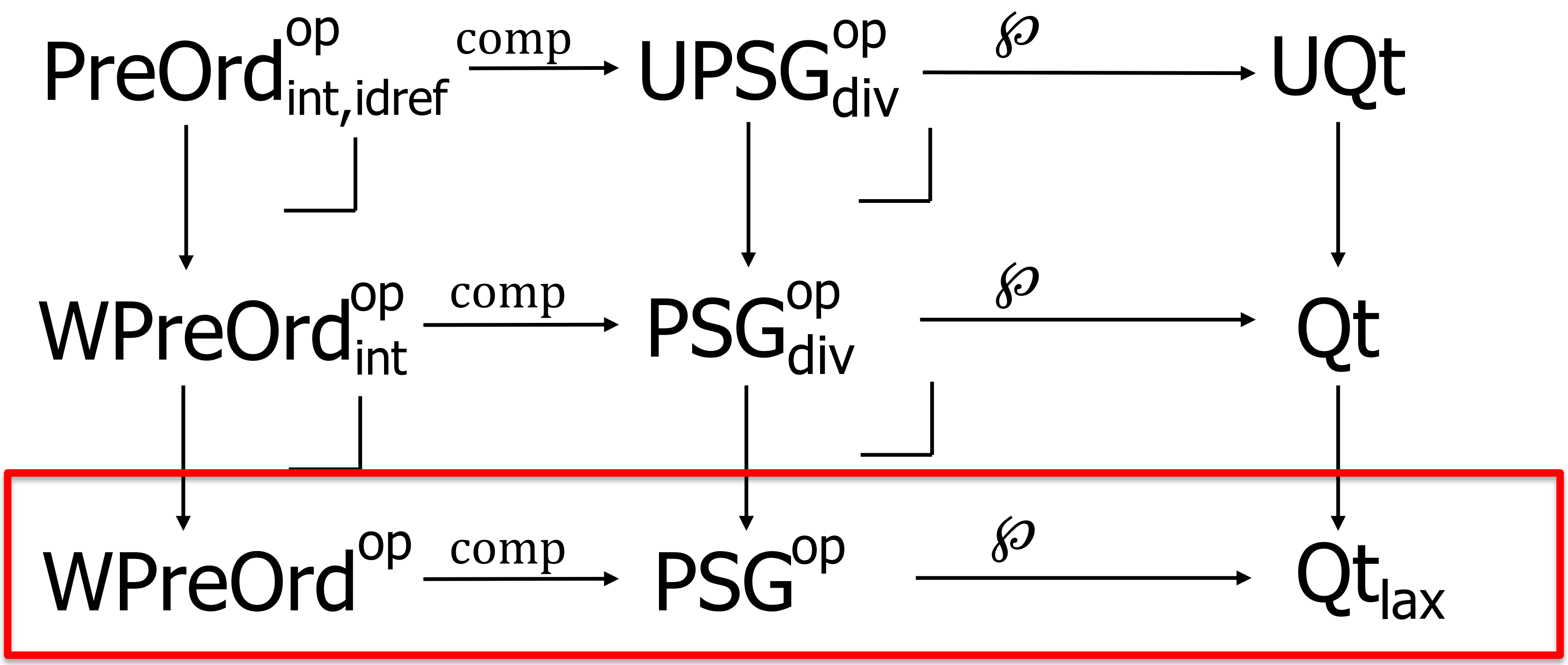
Conditions of arrow
- intermediate value property
- identity reflecting

Conditions of arrow
- dividing

This paper's Results



Construction of quantales from weak preordered sets

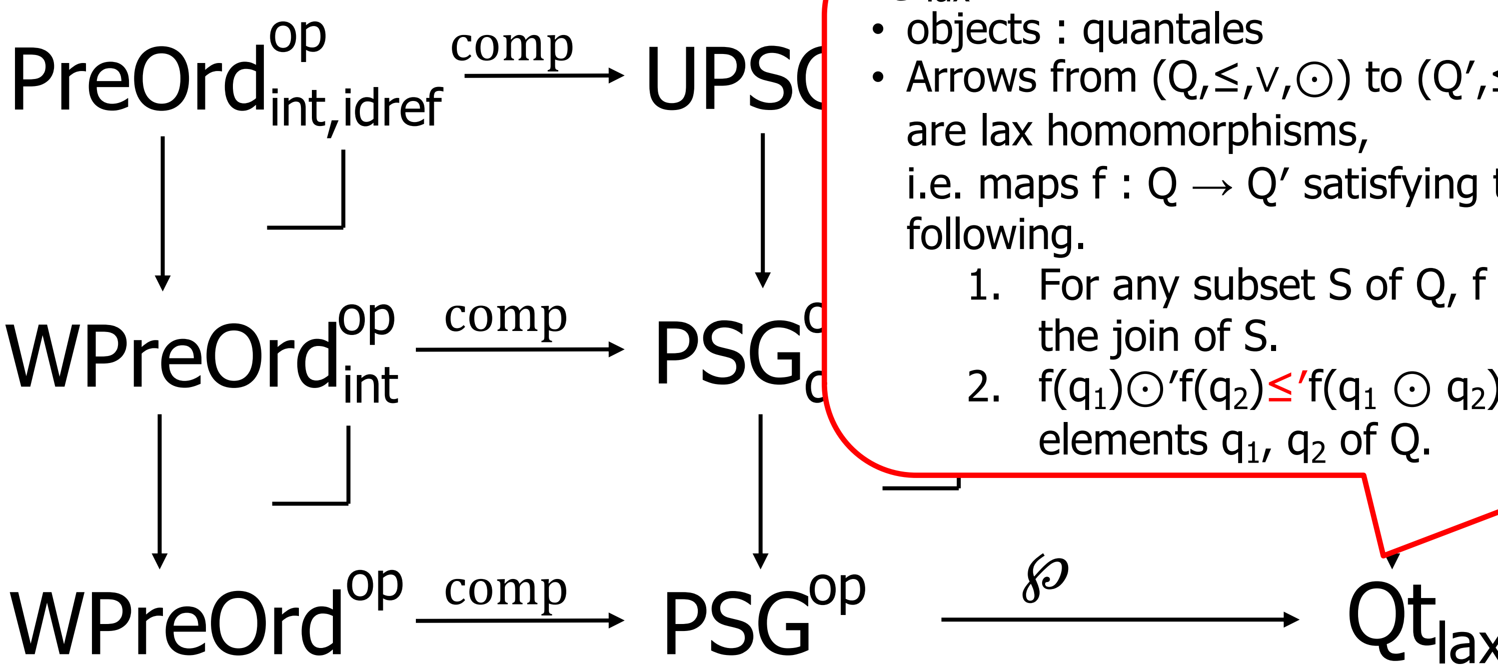


①

Contents

- Functor from the category of preordered sets to the category of the unital quantales
  ① Weak preorder, partial semigroup, quantale lax homomorphism
  ② Intermediate value property, dividing, quantale homomorphism
  ③ Preorder and identity reflecting, unital partial semigroup, unital homomorphism

Construction of quantales from weak preordered sets



Qt$_{lax}$

- objects : quantales
- Arrows from $(Q,\leq,\vee,\odot)$ to $(Q',\leq',\vee',\odot')$ are lax homomorphisms, i.e. maps $f : Q \to Q'$ satisfying the following.
  1. For any subset S of Q, f preserves the join of S.
  2. $f(q_1)\odot' f(q_2) \leq' f(q_1 \odot q_2)$ for each elements q_1, q_2 of Q.

Construction of quantales from weak preordered sets

A partial semigroup is a tuple(X, ·) s.t.

1. X is a set.
2. · is a partial binary function on X (i.e., x · y may be undefined).
3. x · y and (x · y) · z are defined ⇔ y · z and x ·(y · z) are defined.
4. For x,y,z satisfying at least one of the left and right sides (i.e. both) of condition 3, (x · y) · z = x · (y · z).

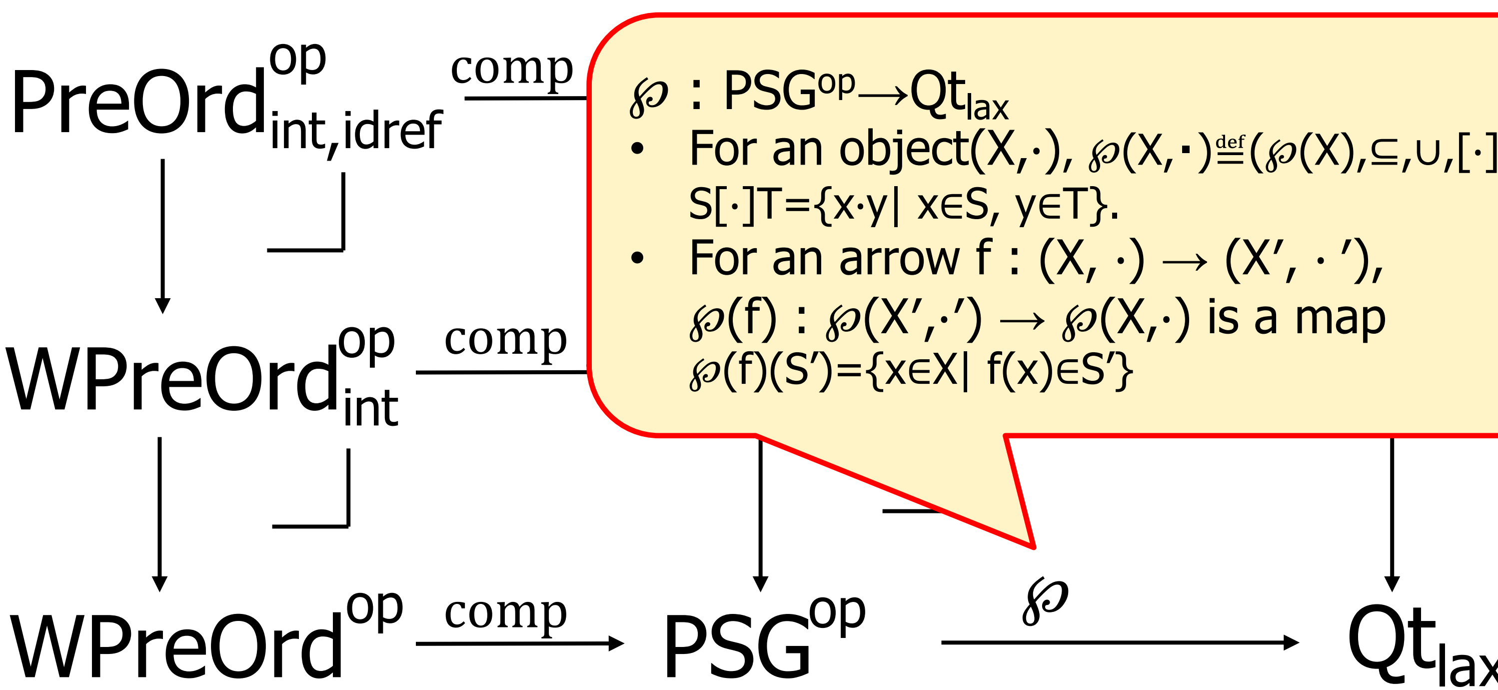
PSG

- objects : partial semigroups
- Arrows from (X,·) to (X',·′) are homomorphisms, i.e. maps f :X →X′ satisfying the following.
    1. f(x) ·′ f(y) is defined for x,y∈ X s.t. x · y is defined
    2. For x,y satisfying condition 1, f(x) ·′ f(y) = f(x · y).

PSG

$$\mathrm{WPreOrd}^{op} \xrightarrow{\mathrm{comp}} \mathrm{PSG}^{op} \xrightarrow{\wp} \mathrm{Qt}_{lax}$$

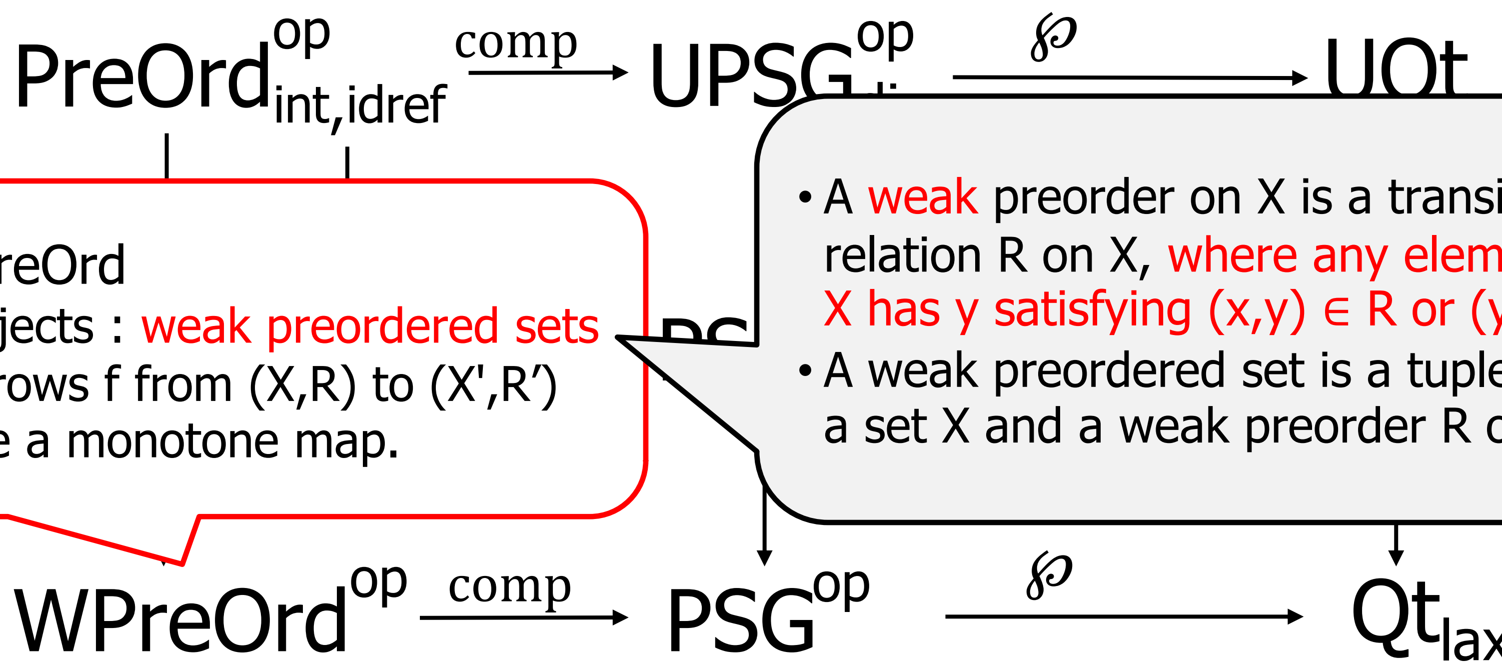
Construction of quantales from weak preordered sets



$\wp$: $PSG^{op} \rightarrow Qt_{lax}$

- For an object $(X,\cdot)$, $\wp(X,\cdot) \stackrel{\mathrm{def}}{=} (\wp(X),\subseteq,\cup,[\cdot])$.
  $S[\cdot]T=\{x\cdot y \mid x\in S,\ y\in T\}$.
- For an arrow $f : (X, \cdot) \rightarrow (X', \cdot')$,
  $\wp(f) : \wp(X',\cdot') \rightarrow \wp(X,\cdot)$ is a map
  $\wp(f)(S')=\{x\in X \mid f(x)\in S'\}$

Construction of quantales from weak preordered sets



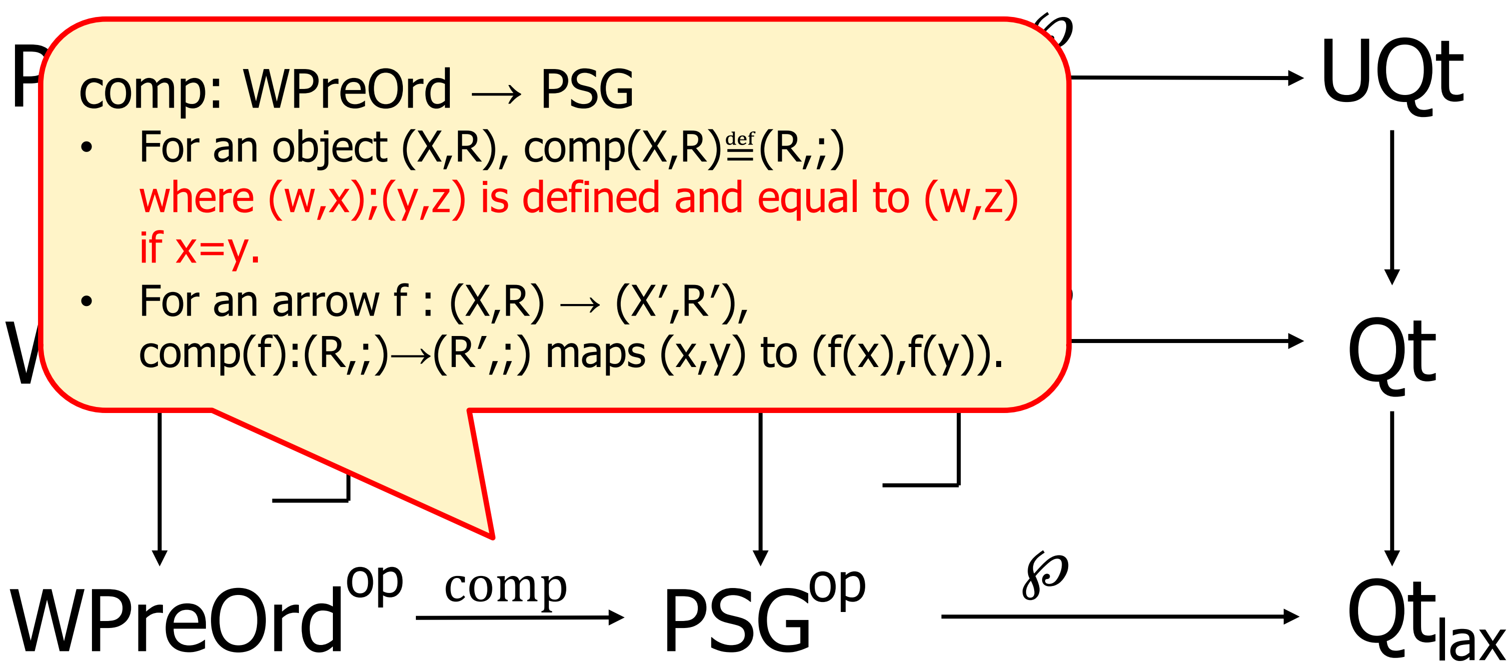
WPreOrd
- objects : weak preordered sets
- Arrows f from (X,R) to (X',R') are a monotone map.

- A weak preorder on X is a transitive relation R on X, where any element x of X has y satisfying (x,y) ∈ R or (y,x) ∈ R.
- A weak preordered set is a tuple(X,R) of a set X and a weak preorder R on X.

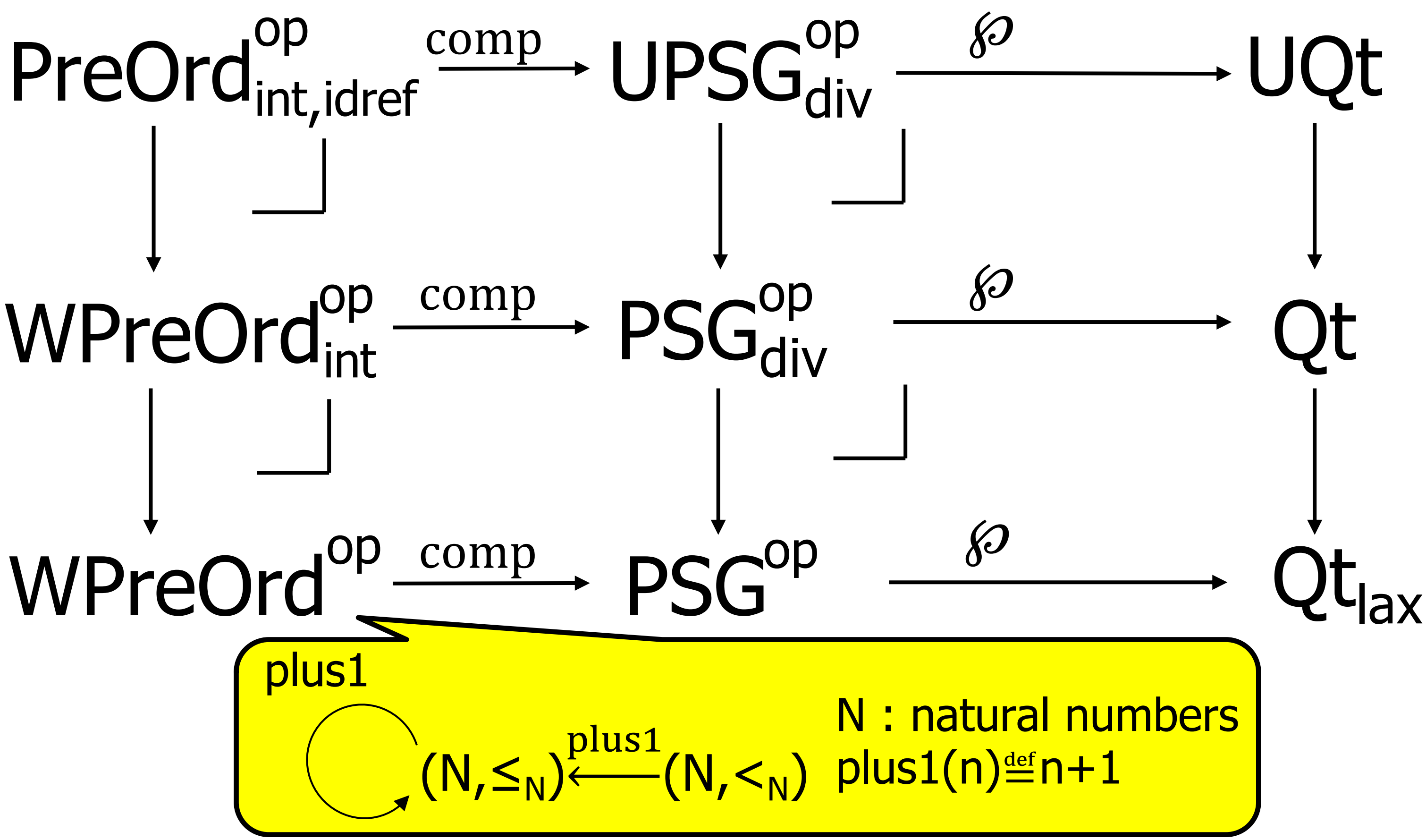
Construction of quantales from weak preordered sets

comp: WPreOrd ⟶ PSG

- For an object (X,R), comp(X,R)$\stackrel{\text{def}}{=}$(R,;) where (w,x);(y,z) is defined and equal to (w,z) if x=y.
- For an arrow f : (X,R) → (X′,R′), comp(f):(R,;)→(R′,;) maps (x,y) to (f(x),f(y)).

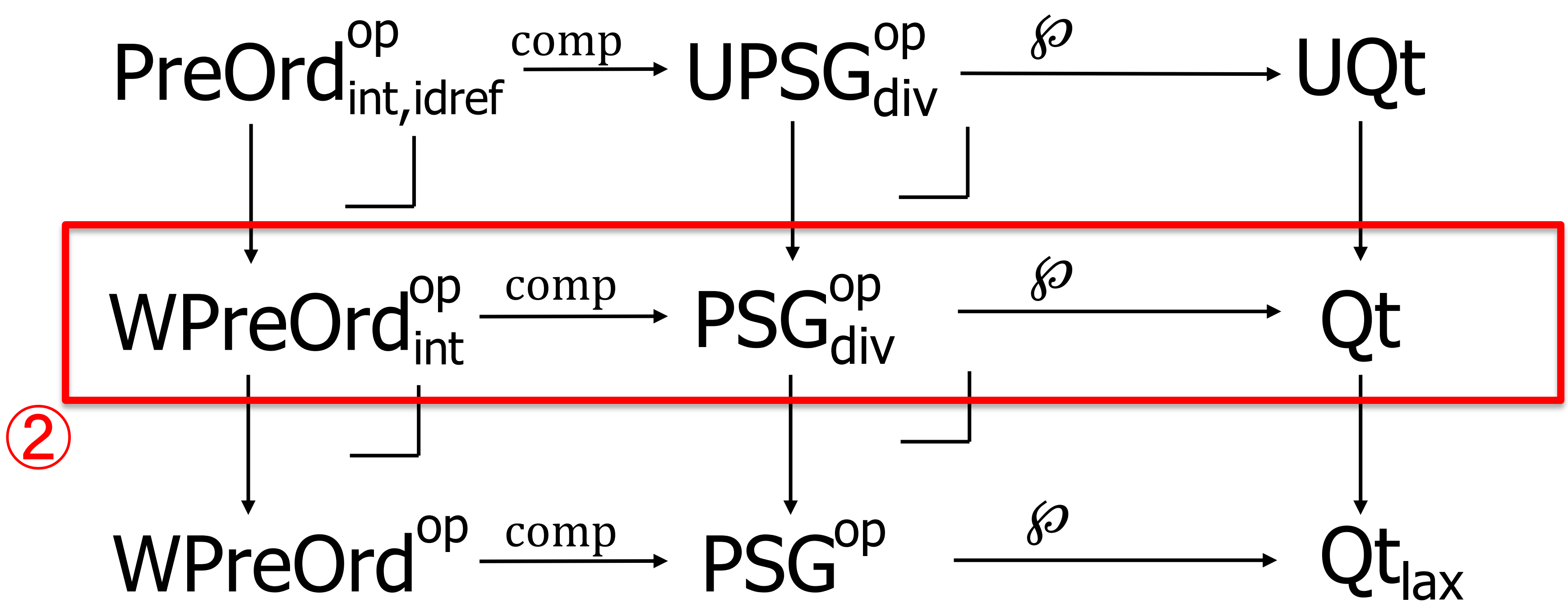


Example1



$PreOrd^{op}_{int,idref}$
comp
$UPSG^{op}_{div}$
℘
UQt
$WPreOrd^{op}_{int}$
comp
PSG^{op}_{div}
℘
Qt
$WPreOrd^{op}$
comp
PSG^{op}
℘
Qt_{lax}
plus1
$(N,\leq_N) \xleftarrow{plus1} (N,<_N)$
N : natural numbers
$plus1(n) \overset{def}{=} n+1$

Conditions corresponding to quantale homomorphism



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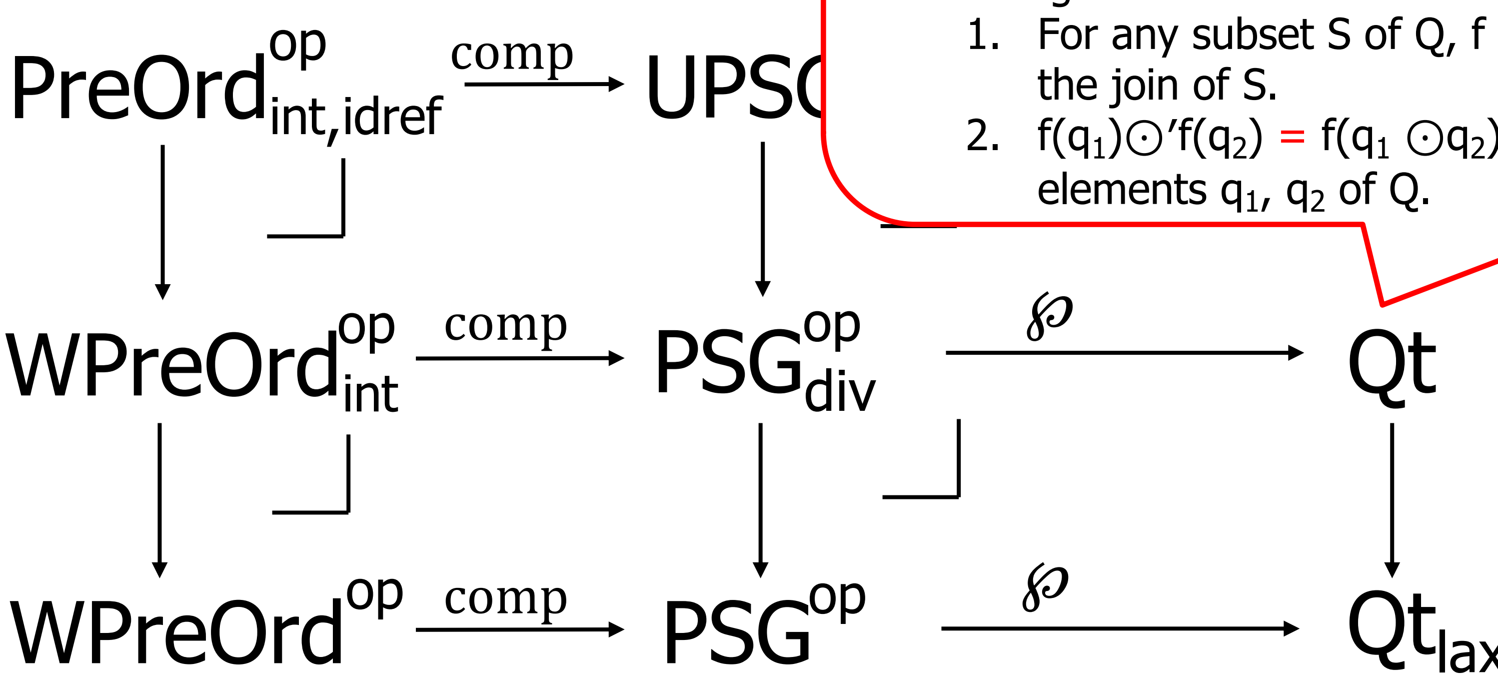
Conditions corresponding to quantale homomorphism

Qt

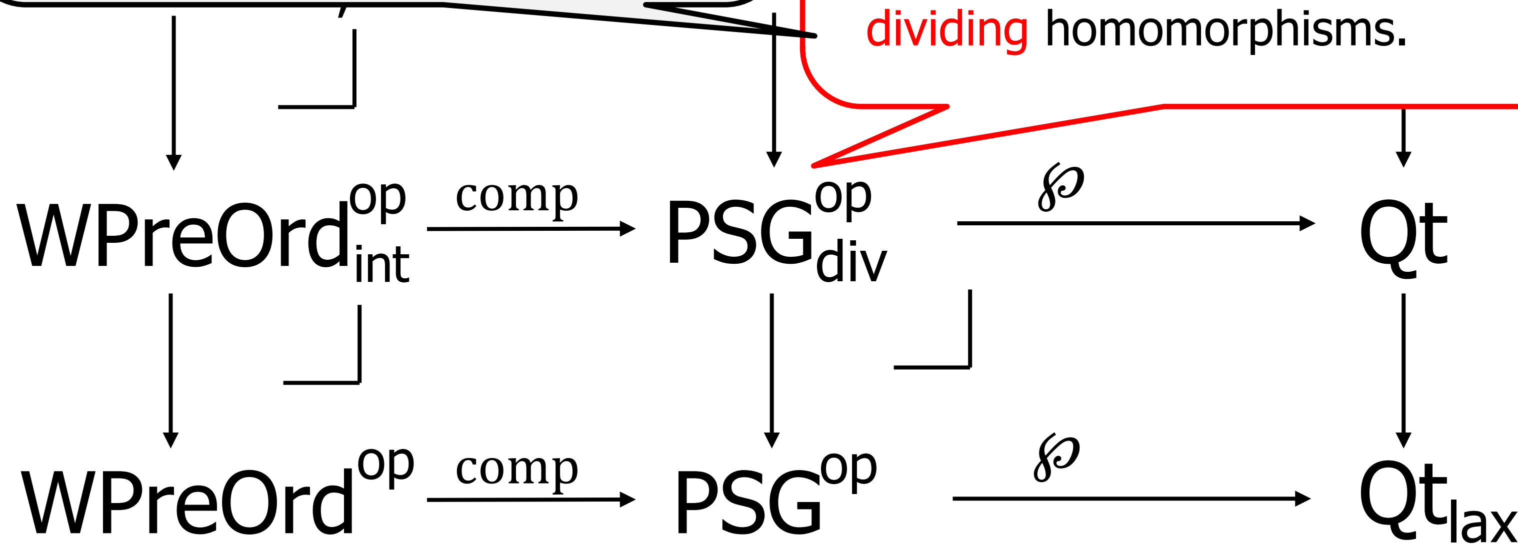
- objects : quantales
- Arrows from $(Q,\leq,\vee,\odot)$ to $(Q',\leq',\vee',\odot')$ are homomorphisms, i.e. maps $f : Q \to Q'$ satisfying the following.
  1. For any subset S of Q, f preserves the join of S.
  2. $f(q_1)\odot' f(q_2) = f(q_1 \odot q_2)$ for each elements q_1, q_2 of Q.



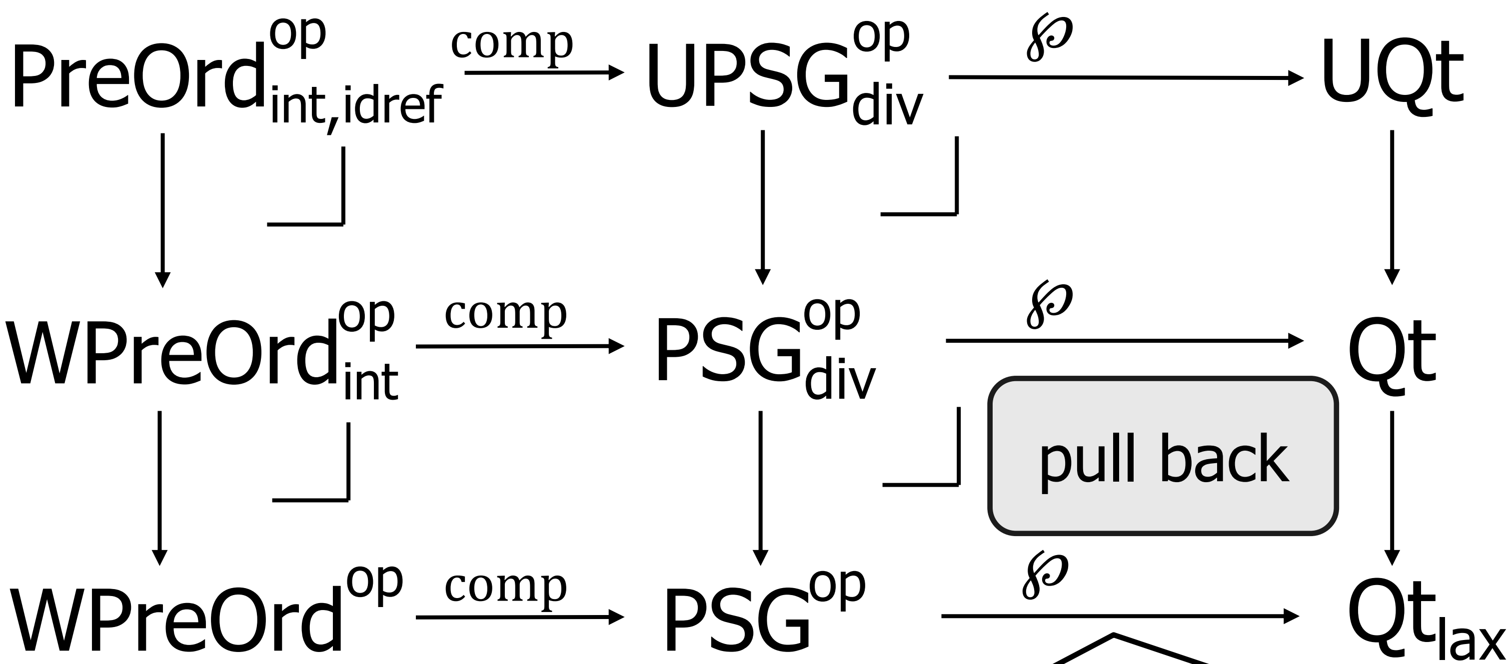
Conditions corresponding to quantale homomorphism

if for each $x', y' \in X'$ and $z \in X$ satisfying $x' \cdot' y' = f(z)$, there exist $x, y \in X$ s.t. $f(x) = x'$, $f(y) = y'$, and $x \cdot y = z$.
(dividing)

$\mathbf{PSG}_{div}$
- objects : partial semigroups
- Arrows f from $(X,\cdot)$ to $(X',\cdot')$ are dividing homomorphisms.



Conditions corresponding to quantale homomorphism



(For any $f \in \mathrm{PSG}^{op}$,) $f \in \mathrm{PSG}^{op}_{div} \Leftrightarrow \wp(f) \in \mathrm{Qt}$

Conditions corresponding to quantale homomorphism

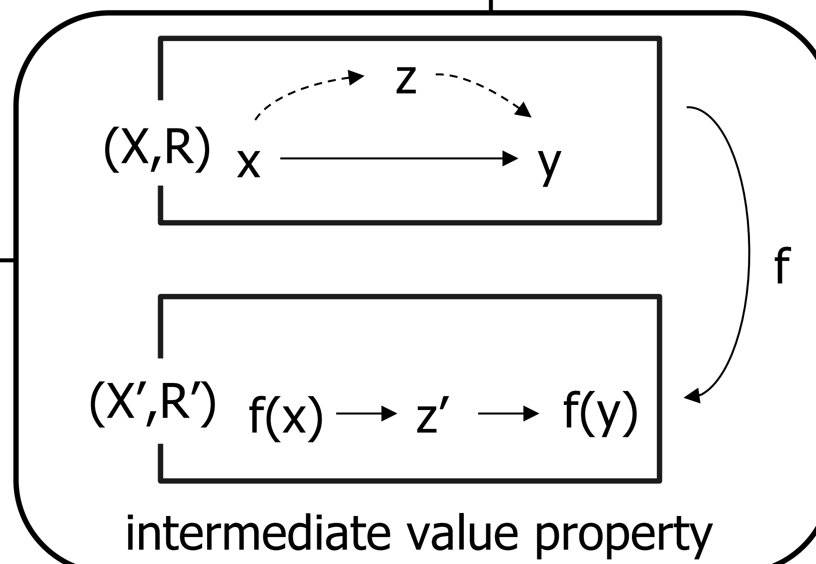
$\mathbf{WPreOrd}_{int}$
- objects : weak preordered sets
- Arrows f from (X, R) to (X′,R′) are monotone maps satisfying the intermediate value property.

if for x, y, z′ s.t. (x, y) ∈ R, (f(x), z′) ∈ R′ and (z′,f(y)) ∈ R′, there exists z ∈ X s.t. f(z) = z′, (x,z) ∈ R, and (z,y) ∈ R.
(the intermediate value property)

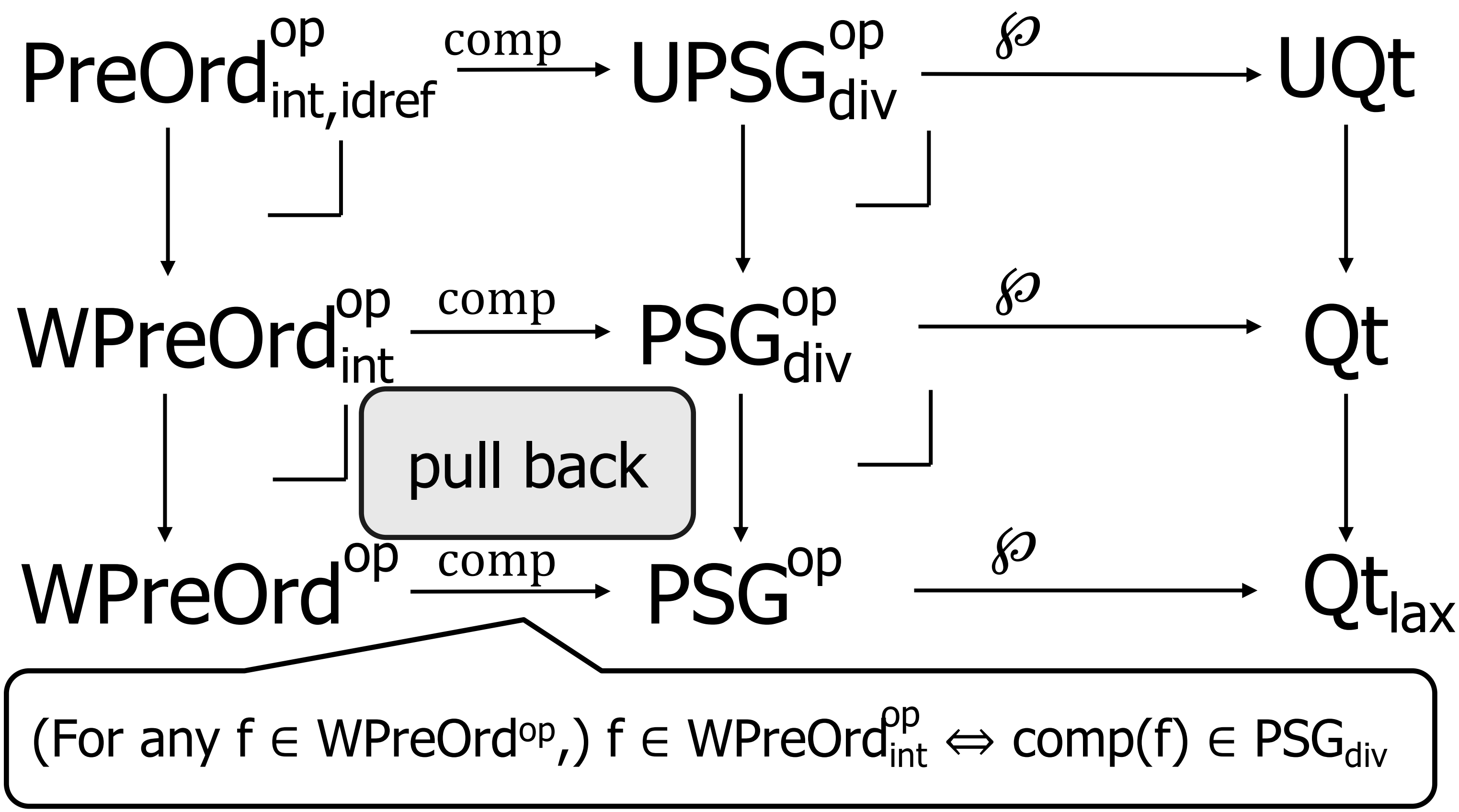
$$\mathrm{UPSG}^{op}$$

$$\mathrm{WPreOrd}^{op}_{int} \xrightarrow{\text{comp}} \mathrm{PSG}^{op}_{div}$$

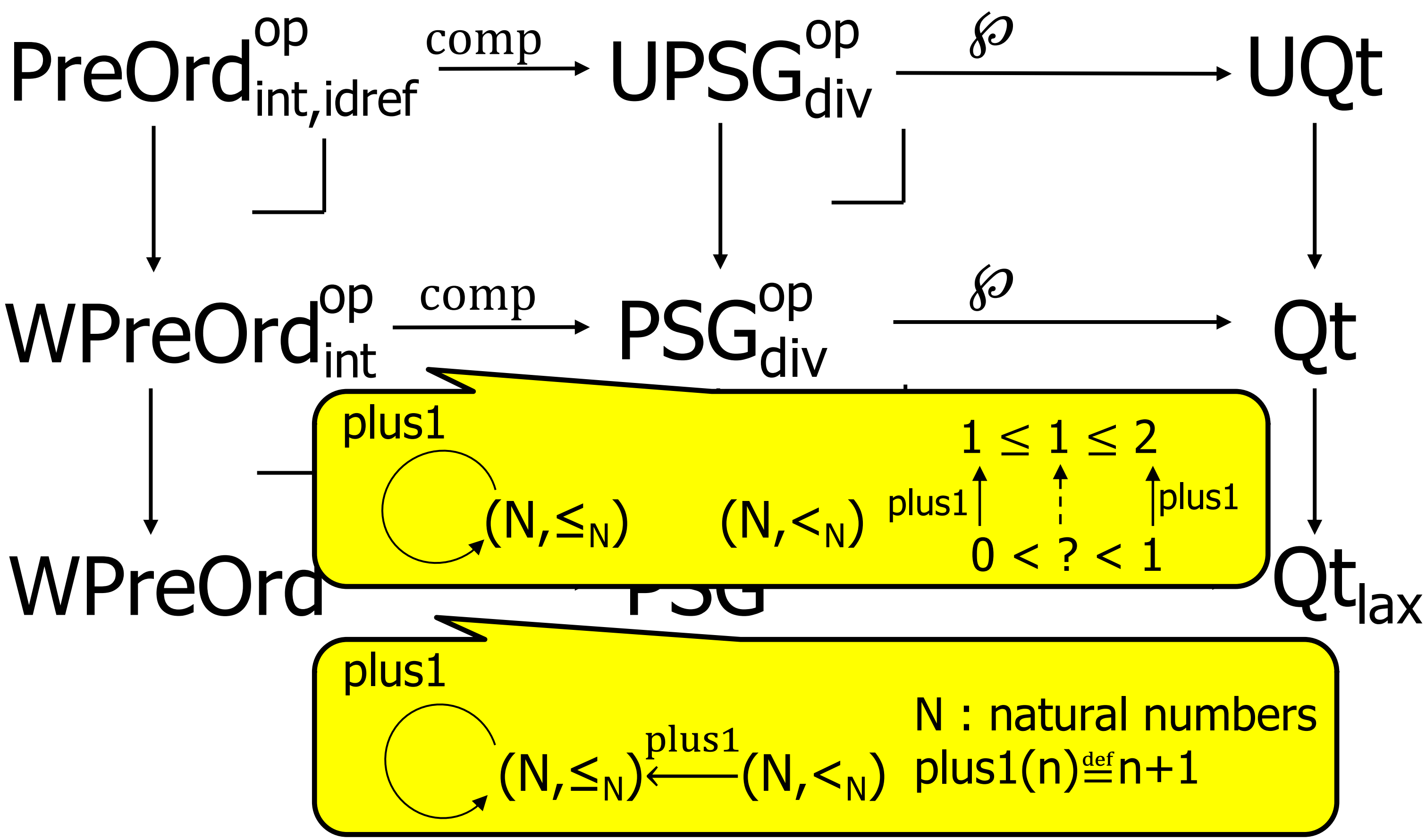
$$\mathrm{WPreOrd}^{op} \xrightarrow{\text{comp}} \mathrm{PSG}^{op}$$



Conditions corresponding to quantale homomorphism



Example2



PreOrd$^{op}_{int,idref}$
comp
UPSG$^{op}_{div}$
℘
UQt
WPreOrd$^{op}_{int}$
comp
PSG$^{op}_{div}$
℘
Qt
WPreOrd
PSG
Qt$_{lax}$
plus1
(N,≤$_N$)
(N,<$_N$)
1 ≤ 1 ≤ 2
plus1
plus1
0 < ? < 1
plus1
(N,≤$_N$)
plus1
(N,<$_N$)
N : natural numbers
plus1(n) $\stackrel{def}{=}$ n+1

Construction of unital quantales from preordered sets

③

$$
\begin{array}{ccccc}
\mathrm{PreOrd}^{op}_{int,idref} & \xrightarrow{\mathrm{comp}} & \mathrm{UPSG}^{op}_{div} & \xrightarrow{\wp} & \mathrm{UQt} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{WPreOrd}^{op}_{int} & \xrightarrow{\mathrm{comp}} & \mathrm{PSG}^{op}_{div} & \xrightarrow{\wp} & \mathrm{Qt} \\
\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\
\mathrm{WPreOrd}^{op} & \xrightarrow{\mathrm{comp}} & \mathrm{PSG}^{op} & \xrightarrow{\wp} & \mathrm{Qt}_{lax}
\end{array}
$$

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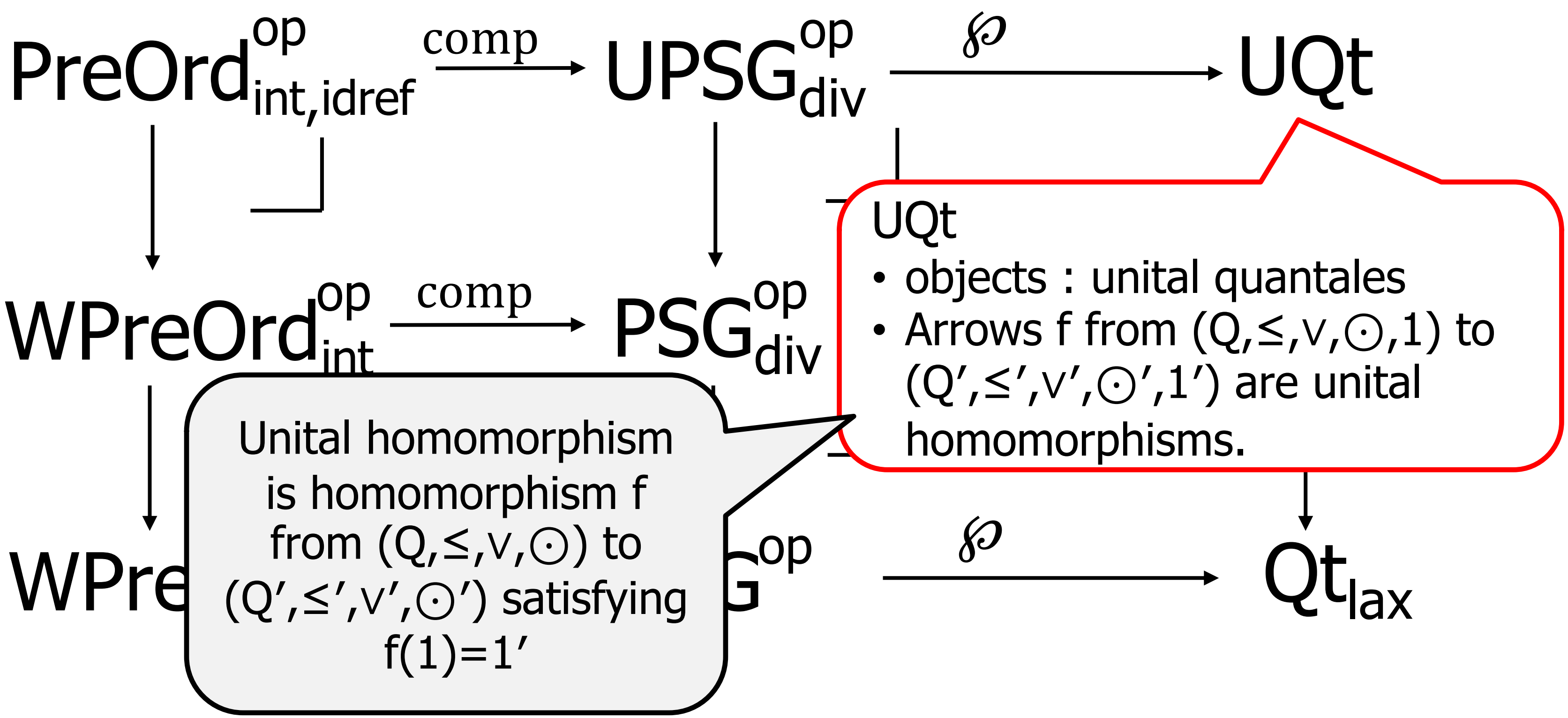
Construction of unital quantales from preordered sets



Unital homomorphism is homomorphism f from (Q,≤,∨,⊙) to (Q′,≤′,∨′,⊙′) satisfying f(1)=1′

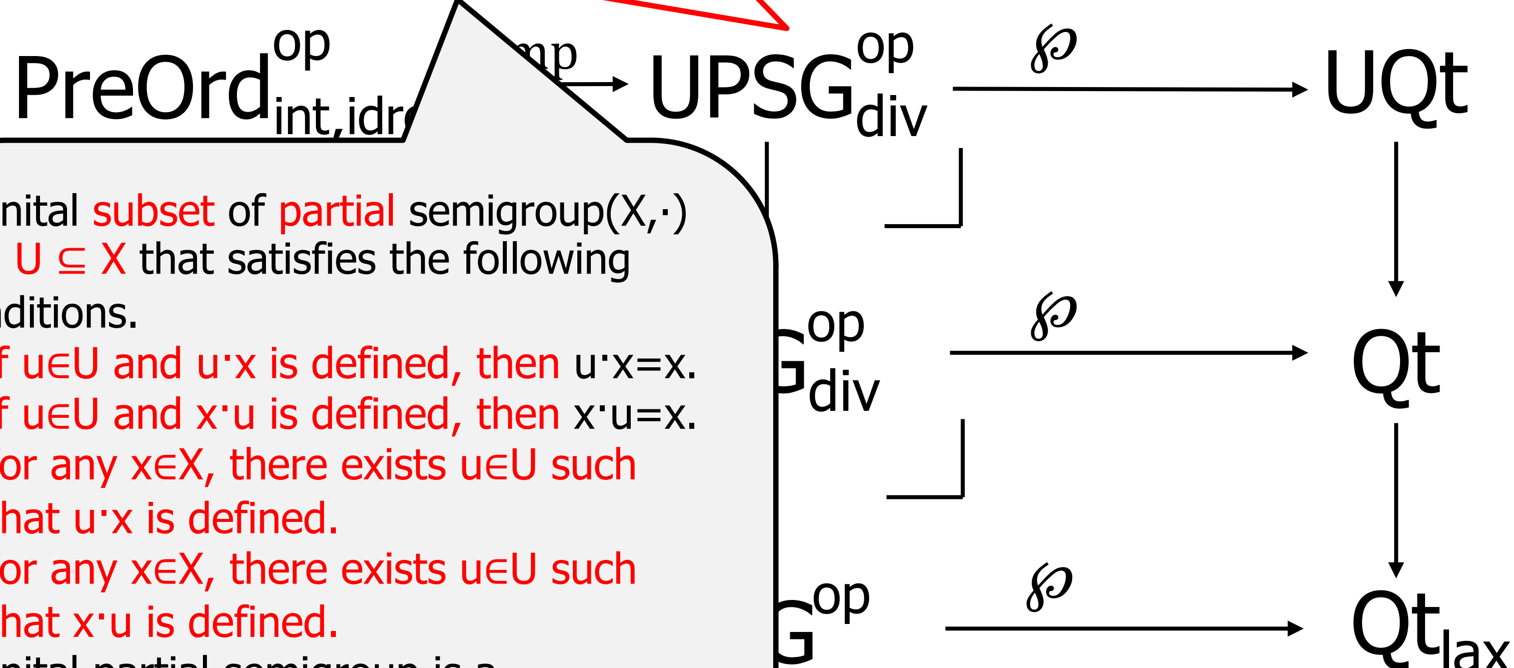
UQt

- objects : unital quantales
- Arrows f from (Q,≤,∨,⊙,1) to (Q′,≤′,∨′,⊙′,1′) are unital homomorphisms.

… quantales from

$UPSG_{div}$

- objects : unital partial semigroups
- Arrows f : (X, · ,U) → (X′, ·′ ,U′) are f : (X, ·) → (X′, ·′) in PSG_{div} s.t. for ∀x∈X, x∈U ⇔ f(x)∈U′.



A unital subset of partial semigroup(X,·) is a U ⊆ X that satisfies the following conditions.

1. if u∈U and u·x is defined, then u·x=x.
2. if u∈U and x·u is defined, then x·u=x.
3. for any x∈X, there exists u∈U such that u·x is defined.
4. for any x∈X, there exists u∈U such that x·u is defined.

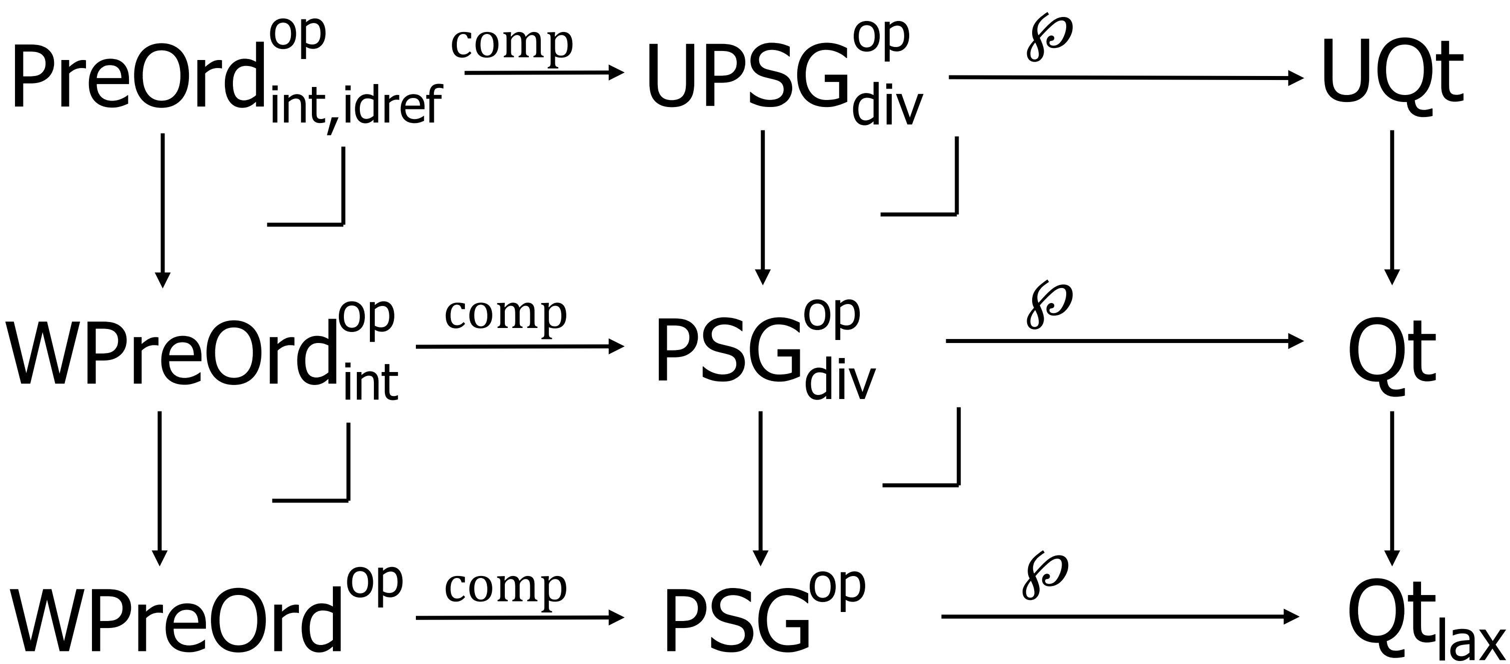
A unital partial semigroup is a tuple(X,·,U) that satisfies the following.

- (X,·) is a partial semigroup,
- U is the unital subset of (X,·).

Co… ales from pre…

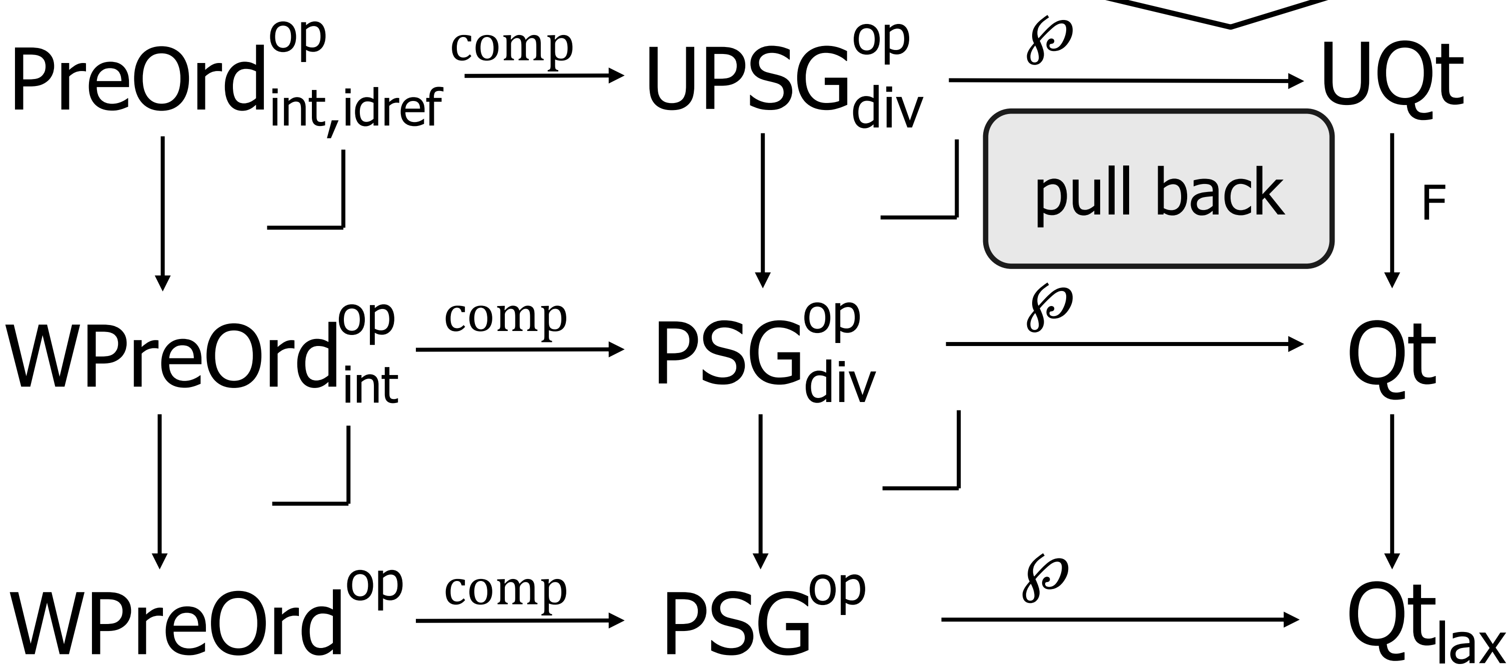
$\wp : UPSG_{div} \rightarrow UQt$

- For an object $(X,\cdot,U)$, $\wp(X,\cdot,U) \overset{\text{def}}{=} (\wp(X),\subseteq,\cup,[\cdot],U)$.
- For an arrow $f : (X,\cdot,U) \rightarrow (X',\cdot',U')$, $\wp(f) : \wp(X',\cdot',U') \rightarrow \wp(X,\cdot,U)$ is a map $\wp(f)(S')=\{x\in X \mid f(x)\in S'\}$.



Construction of unital quantales from preordered sets

$(X, \cdot, U)\in \mathrm{UPSG}^{op}_{div} \Leftrightarrow (X, \cdot)\in \mathrm{PSG}^{op}_{div}$ and $(Q,U)\in \mathrm{UQt}$ s.t. $\wp(X,\cdot)=Q=F(Q,U)$

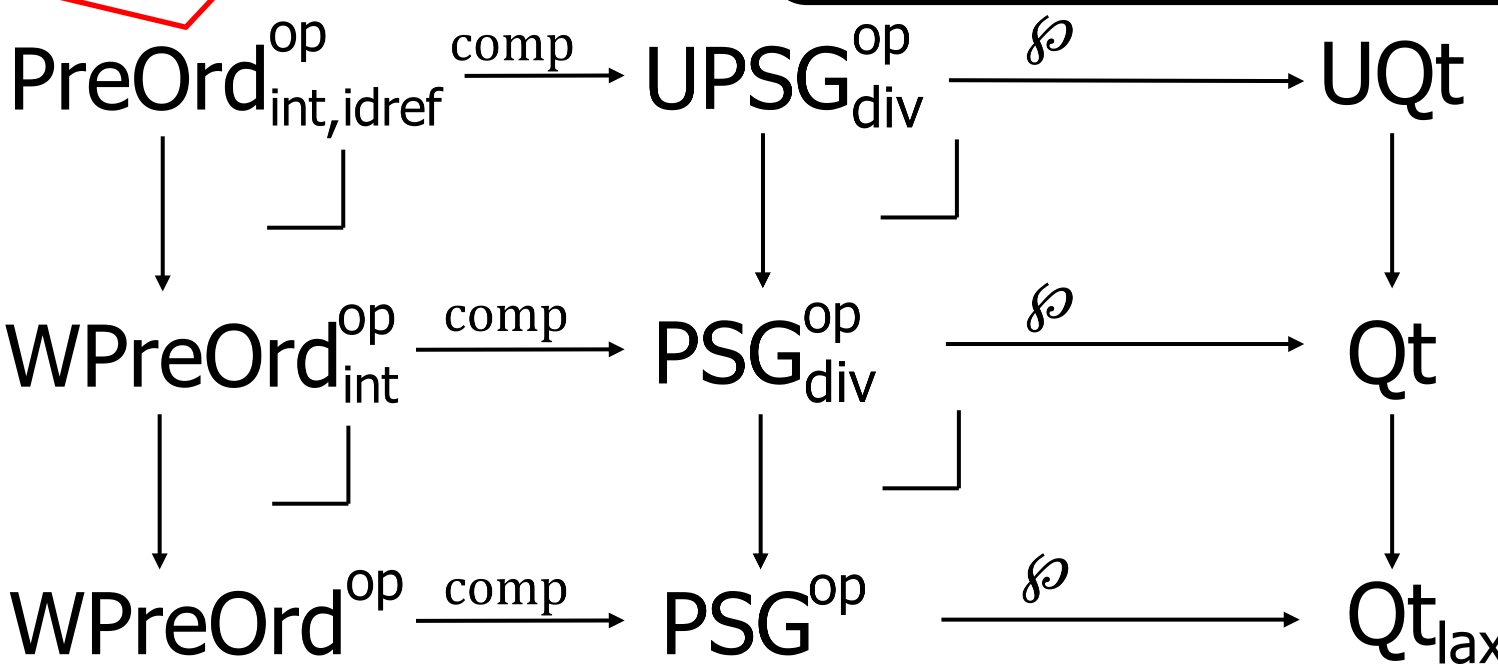


nital quantales from

$PreOrd_{int,idref}$

- objects : preordered sets
- Arrows f from (X,≤) to (X′,≤′) are monotone maps satisfying the intermediate value property and Id-reflecting.

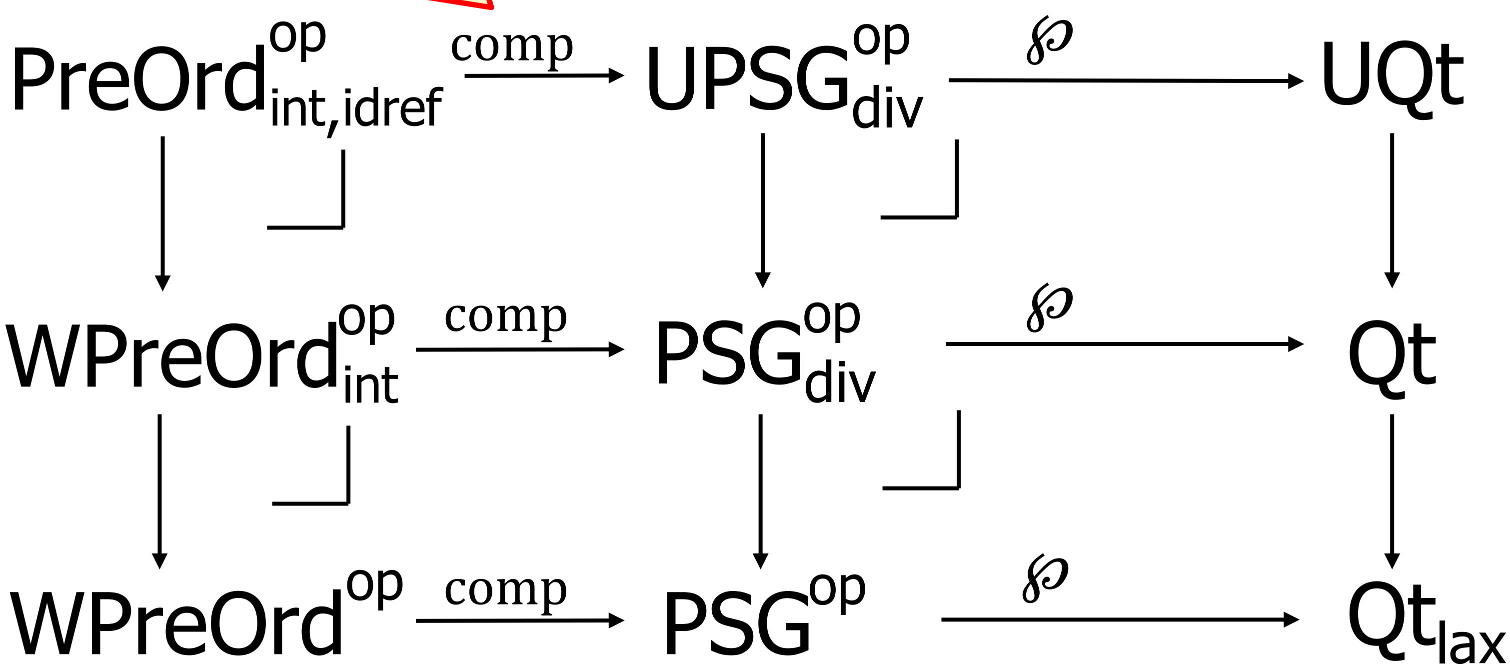
Id-refrecting means that if x≤y and f(x)=f(y), then x=y.



les from

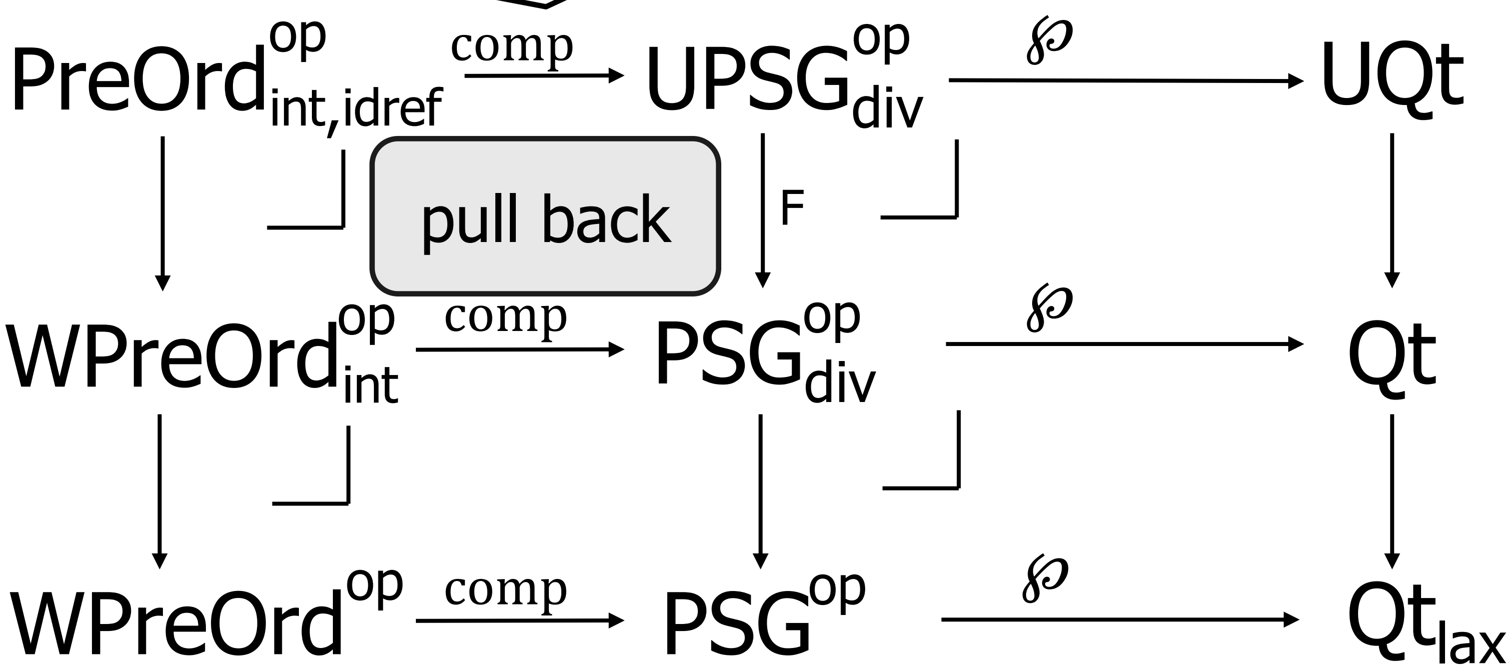
comp: $\mathrm{PreOrd}_{\mathrm{int,idref}} \to \mathrm{UPSG}_{\mathrm{div}}$

- For an object$(X,\leq)$, comp$(X,\leq) \stackrel{\text{def}}{=} (\leq;;;\Delta_X)$.
- For an arrow $f : (X,\leq) \to (X',\leq')$, comp$(f)$: $(\leq;;;\Delta_X) \to (\leq';;;\Delta_X')$ maps (x,y) to $(f(x),f(y))$.

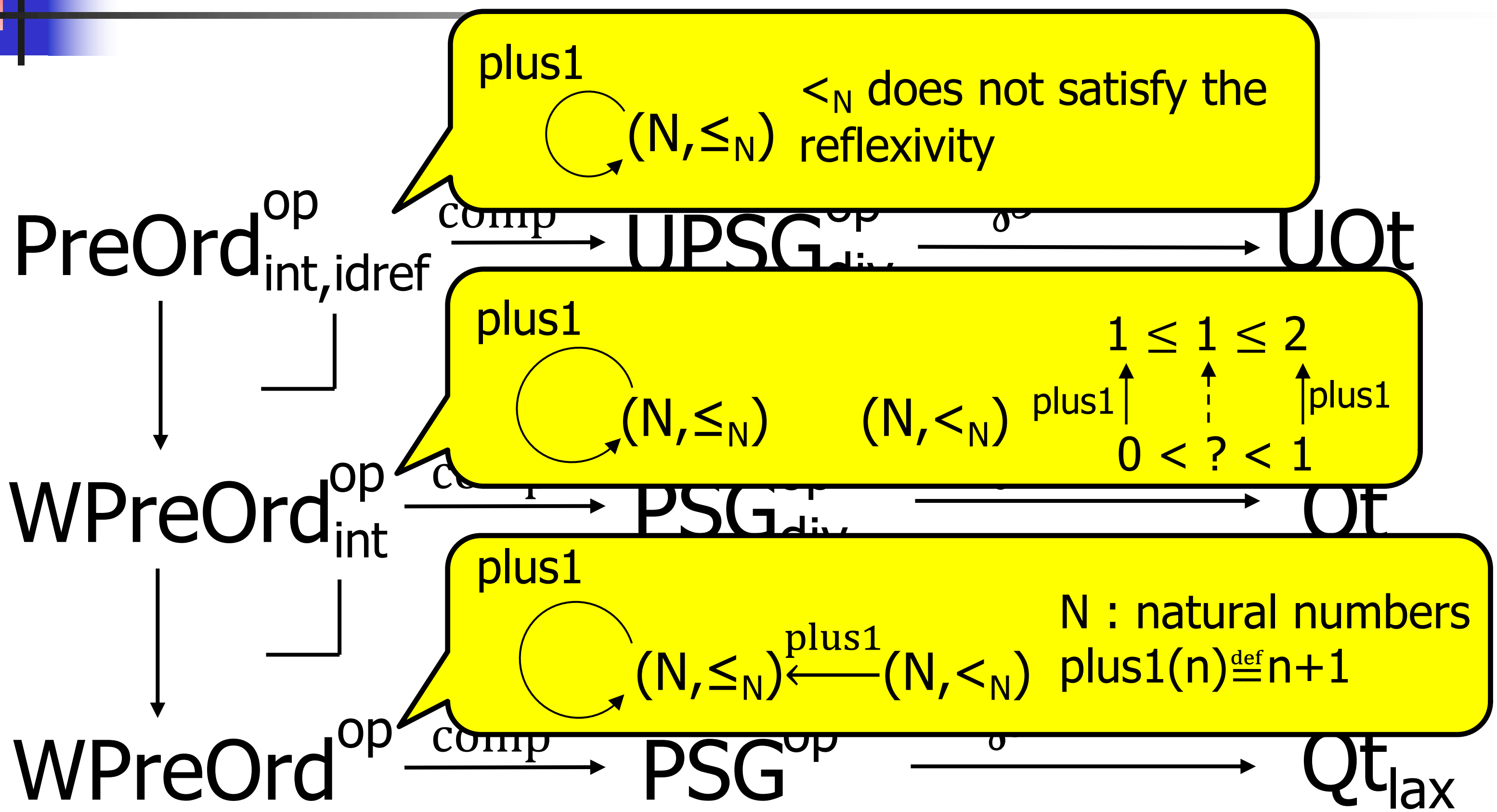


Construction of unital quantales from preordered sets

$(X,R)\in \mathrm{PreOrd}^{op}_{int,idref} \Leftrightarrow (X,R)\in \mathrm{WPreOrd}^{op}_{int}$ and $(Y,\cdot,U) \in \mathrm{UPSG}_{div}$ s.t. $\mathrm{comp}(X,R)=(R,;)=(Y,\cdot)=F(Y,\cdot,U)$



Example3



PreOrd op int,idref
UPSG op div
UOt
WPreOrd op int
PSG op div
Ot
WPreOrd op
PSG op
Qt lax
plus1
(N,≤N)
<N does not satisfy the reflexivity
plus1
(N,≤N)
(N,<N)
1 ≤ 1 ≤ 2
plus1
plus1
0 < ? < 1
plus1
(N,≤N)
plus1
(N,<N)
N : natural numbers
plus1(n) def= n+1

Conclusion and Future Work

- We defined the functors comp and $\wp$ between categories using the following:
  ① Weak preorder, partial semigroup, quantale lax homomorphism
  ② Intermediate value property, dividing, quantale homomorphism
  ③ Preorder and identity reflecting, unital partial semigroup, unital homomorphism

- Future Work
  - To find a functor from UQt to $\mathrm{PreOrd}^{op}_{int,idref}$
    - This paper includes a functor suff to $\mathrm{PreOrd}^{op}_{int,idref}$, which may be useful.