Commutative doubly-idempotent semirings determined by chains and by preorder forests

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Commutative doubly-idempotent semirings

Definition

Semiring

A **semiring A** $= (A, +, 0, \cdot, 1)$ is defined by the following equations for all $x, y, z \in A$:

$$(x + y) + z = x + (y + z) \qquad (xy)z = x(yz)$$

$$x + y = y + x \qquad x1 = x = 1x$$

$$x + 0 = x \qquad x0 = 0 = 0x$$

$$x(y + z) = xy + xz \qquad (x + y)z = xz + yz$$

- An **idempotent semiring** is a semiring **A** such that $x + x = x$.
- In this case we replace $+$ with $\vee$ since $(A, \vee)$ is a semilattice.
- **A** is **doubly idempotent** if $x \vee x = x$ and $xx = x$.
- **A** is **commutative** if $xy = yx$.

Alternative definition using semilattices

Semilattice with 0

A **semilattice with 0** is an algebra $(A, \vee, 0)$ such that $\vee$ is associative, commutative, idempotent ($x \vee x = x$), and $x \vee 0 = x$.

A semilattice is **partially ordered** by $x \le y \iff x \vee y = y$.

A **commutative doubly-idempotent semiring** (**cdi-semiring**) is of the form $(A, \vee, 0, \cdot, 1)$ such that:

- $(A, \vee, 0)$ is a semilattice with 0 (ordered by $\le$)
- $(A, \cdot, 1)$ is a semilattice with 1 (ordered by $x \sqsubseteq y \Leftrightarrow xy = x$)
- $x0 = 0$, and $x(y \vee z) = xy \vee xz$ hold for all $x, y, z \in A$.

Example

All **bounded distributive lattices** are cdi-semirings where xy is the meet (greatest lower bound) of x and y.

In this case $x \le y$ if and only if $x \sqsubseteq y$.

We look at **three subclasses** of cdi-semirings

Why look at **restricted classes** of cdi-semirings?

- While distributive lattices are well understood, the class of cdi-semirings is **much bigger**. There is no general structure theory.
- The class of cdi-semirings is defined by a set of identities, hence it is a variety.
- Chajda and Länger [2017] proved cdi-semirings are the smallest variety containing all bounded distributive lattices and $\mathbf{S}_3$, a 3-element cdi-semiring that is not a distributive lattice.

Consider the number of algebras for each size (up to isomorphism)

# of elements =	1	2	3	4	5	6	7	8
# of cdi-semirings	1	1	2	6	20	77	333	1589
# of distr. lattices	1	1	1	2	3	5	8	15

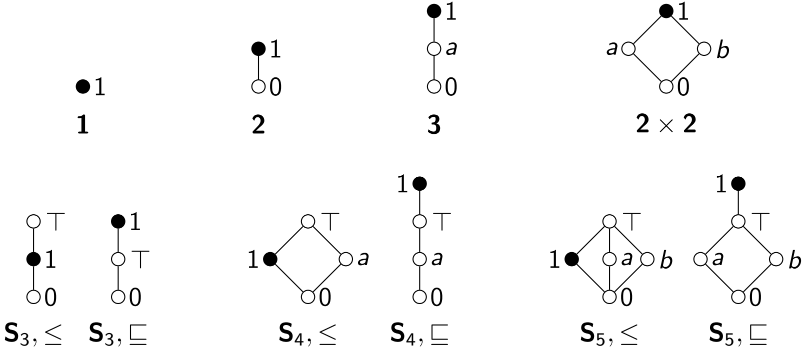
Part 1: Seven cdi-semirings of height ≤ 2

The **height** of a join-semilattice is the length of the longest chain in it.

With the restriction on height of cdi-semirings to be less or equal to two for $(A, \vee)$, we have the following theorem:

Theorem one

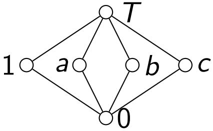
There are, up to isomorphism, **seven** cdi-semirings of height ≤ 2.



Seven cdi-semirings

Proof idea:

In order to prove the theorem, one would want to show that we cannot have a cdi-semirings of the following shape of height 2.



And then show the same for every lattice of height two with more than 3 atoms.

Catalan semiring

A **Catalan semiring** is a multiplicatively linear cdi-semiring, i.e. $x \sqsubseteq y$ or $y \sqsubseteq x$ for all x, y. Hence the multiplicative order is a **chain**.

For $\mathbf{A}$ and $\mathbf{B}$ Catalan semirings, we define the **Catalan sum** $\mathbf{C} = \mathbf{A} \copyright \mathbf{B}$ to be the structure over the disjoint union of $\mathbf{A}$ and $\mathbf{B}$.

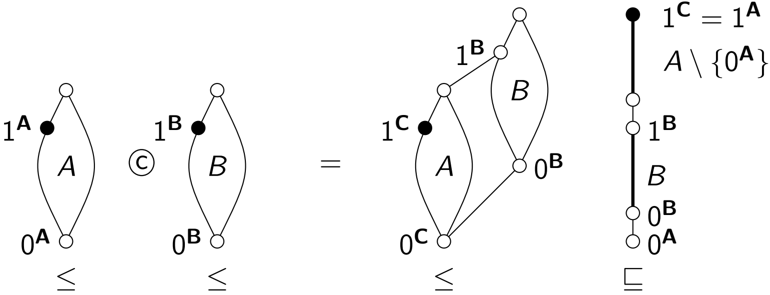
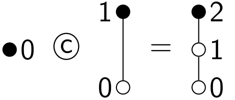
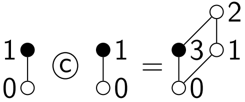


Figure: The Catalan sum $\mathbf{C} = \mathbf{A} \copyright \mathbf{B}$, where $0, 1$ of $\mathbf{C}$ are $0^{\mathbf{A}}, 1^{\mathbf{A}}$.

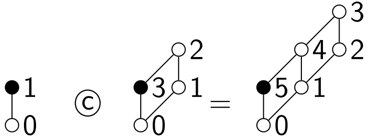
Consider the following examples:



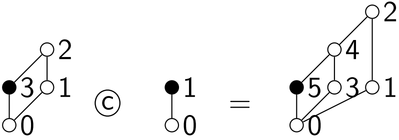
$\mathbf{C}_1^1$ $\mathbf{C}_1^2$ $\mathbf{C}_1^3$



$\mathbf{C}_1^2$ $\mathbf{C}_1^2$ $\mathbf{C}_3^4$



$\mathbf{C}_1^2$ $\mathbf{C}_4^3$ $\mathbf{C}_{17}^6$



$\mathbf{C}_4^3$ $\mathbf{C}_1^2$ $\mathbf{C}_{26}^6$

Note that the Catalan sum in not commutative

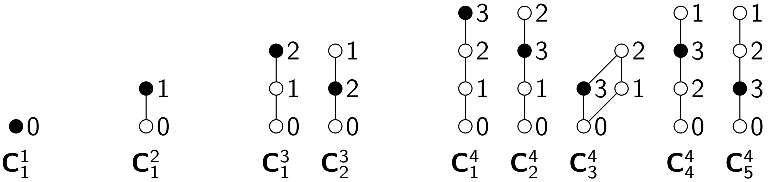
The number of Catalan semirings

Now we replace the restriction on height, with a restriction that $(A, \cdot)$ must be multiplicatively linear.

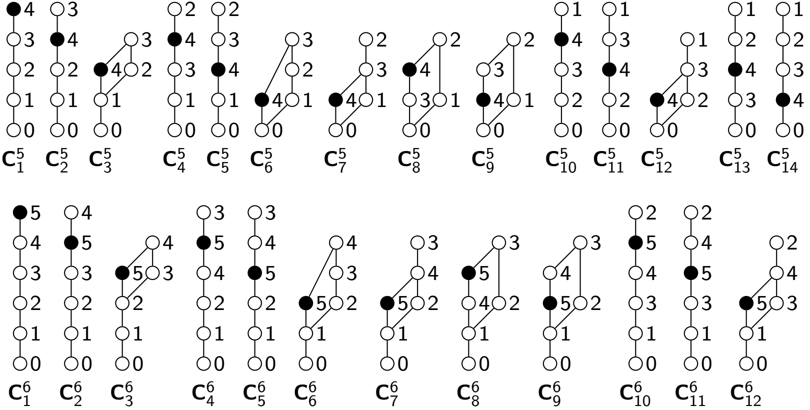
Theorem two

The number of Catalan semirings with $n+1$ elements, up to isomorphism, is the n^{th} Catalan number

$$C_n = \frac{1}{n+1}\binom{2n}{n} = 1, 1, 2, 5, 14, 42, \ldots$$



Catalan semirings of size 5 and some of size 6



Comparing Catalan semirings with all cdi-semirings

# of elements =	1	2	3	4	5	6	7	8
# of distr. lattices	1	1	1	2	3	5	8	15
# of cdi-semirings	1	1	2	6	20	77	333	1589
# of Catalan semirings	1	1	2	5	14	42	132	429

Part 3: Boolean cdi-semirings

Finally, we restrict $\mathbf{B} = (B, \vee, 0)$ to be a Boolean semilattice.

- An **atom** in $\mathbf{B}$ is an element $a \neq 0$ such that $x < a$ implies $x = 0$.
- Denote the set of atoms of $\mathbf{B}$ by $At(\mathbf{B})$.
- $\mathbf{B}$ is **atomic** if every nonzero element has some atom below it.

Boolean cdi-semirings and Quantales

Complete semilattice and lattice

A $\vee$-**semilattice is complete** if $\bigvee S$ (sup) exist for all $S \subseteq B$.

A **lattice is complete** if $\bigwedge S$ (inf) and $\bigvee S$ exist for all $S \subseteq B$.

Quantales

A **nonassociative quantale B** $= (B, \bigvee, \cdot)$ is a complete join-semilattice $(B, \bigvee)$ with a binary operation $\cdot$ such that $x(\bigvee Y) = \bigvee_{y \in Y} xy$ and $(\bigvee Y)x = \bigvee_{y \in Y} yx$ for all $x \in B$ and $Y \subseteq B$.

The lemma in the following slide shows that the results work for **nonassociative nonunital complete atomic Boolean idempotent semirings**, also known as **nonassociative atomic Boolean quantales**.

Boolean nonassociative quantales

Ternary relation

For a nonassociative atomic Boolean quantale **B**, define the ternary relation R on the atoms of **B** by

$$R(x, y, z) \iff x \leq yz$$

Lemma

1. *Define a ternary relation* $R \subseteq A^3$ *by* $R(x, y, z) \iff x \leq yz$. *Then for all* $b, c \in B$,

$$bc = \bigvee \{x : R(x, y, z) \text{ for some } y \leq b, z \leq c\}.$$

2. *Suppose* $R \subseteq A^3$ *is a ternary relation on a set* A, *and define* $\mathbf{B} = (\mathcal{P}(A), \bigcup, \cdot)$ *where for* $Y, Z \in \mathcal{P}(A)$

$$Y \cdot Z = \{x : R(x, y, z) \text{ for some } y \in Y, z \in Z\}.$$

Then **B** *is a nonassociative atomic Boolean quantale.*

We split associativity into the two inequalities:

subassociativity $(xy)z \leq x(yz)$

supassociativity $(xy)z \geq x(yz)$.

Theorem

Let $\mathbf{B}$ *be a nonassociative atomic Boolean quantale with* R *defined on* $A = At(\mathbf{B})$ *as on previous slide. Then* $\mathbf{B}$ *is:*

mult. idempot. $\Leftrightarrow$ $R(x,x,x) \,\&\, (R(x,y,z) \Rightarrow x = y \text{ or } x = z)$

subassociative $\Leftrightarrow$ $(R(u,x,y) \,\&\, R(w,u,z) \Rightarrow \exists v (R(v,y,z) \,\&\, R(w,x,v)))$

right unital $\Leftrightarrow$ $\exists I \subseteq A (x = y \Leftrightarrow \exists z \in I \, R(x,y,z))$

left unital $\Leftrightarrow$ $\exists I \subseteq A (x = z \Leftrightarrow \exists y \in I \, R(x,y,z))$

Ternary relation R, and the reflexive relations P and Q

Some definitions

R is said to be **commutative** if

$$R(x, y, z) \iff R(x, z, y)$$

- R is commutative $\iff$ the corresponding quantale is commutative.

We now observe that if multiplication is idempotent, then the ternary relation can be replaced by two **reflexive relations** P **and** Q corresponding to each case of the formula

$$R(x, y, z) \Rightarrow x = y \text{ or } x = z$$

We will define P and Q in the next slide.

Ternary relations and reflexive relations

Let define P and Q, and relate them to R

Lemma

An idempotent ternary relation $R \subseteq A^3$ is definitionally equivalent to a pair of reflexive binary relations $P, Q \subseteq A^2$ via the following definitions definitions.

Defining P, Q from R:

$$P(x, y) \Leftrightarrow R(x, y, x) \qquad Q(x, y) \Leftrightarrow R(x, x, y)$$

Defining R from P, Q:

$$R(x, y, z) \Leftrightarrow (x = y \;\&\; Q(y, z)) \text{ or } (x = z \;\&\; P(z, y)).$$

- R is commutative if and only if $P = Q$. In this case the structure of nonassociative Boolean cdi-semirings is determined by Q.

Ternary relations and reflexive relations

A simplified version of the previous lemma

Theorem

An idempotent ternary relation $R \subseteq A^3$ is subassociative if and only if the corresponding reflexive relations P, Q satisfy

$$
\begin{array}{lll}
(\mathrm{P}_1) & P(x,y)\ \&\ P(y,z) \Rightarrow P(x,z) & P\text{-transitivity} \\
(\mathrm{P}_2) & Q(x,y)\ \&\ Q(x,z) \Rightarrow Q(y,z) \text{ or } P(z,y) & PQ\text{-forest} \\
(\mathrm{P}_3) & P(x,y)\ \&\ Q(y,z)\ \&\ x \neq y \Rightarrow P(x,z) &
\end{array}
$$

To characterize supassociativity of R, it suffices to interchange P, Q in these conditions to obtain (P'_1), (P'_2), (P'_3).

Hence R is associative $\iff$ P, Q satisfy all six conditions $\iff$ $\mathbf{B} = (\mathcal{P}(A), \bigcup, \cdot)$ is an (associative) atomic Boolean quantale.

The commutative case: Preorder forests

- A **preorder** is a reflexive transitive binary relation,
- A **partial order** is a preorder that is antisymmetric:

$$P(x, y) \ \& \ P(y, x) \Rightarrow x = y$$

- A **forest** is a partial order such that

$$P(x, y) \ \& \ P(x, z) \Rightarrow P(y, z) \text{ or } P(z, y)$$

i.e., all the elements above a given element are linearly ordered.

- A **preorder-forest** is a preorder order such that

$$P(x, y) \ \& \ P(x, z) \Rightarrow P(y, z) \text{ or } P(z, y)$$

- A forest can have many connected components, each is a **tree**.
- If each tree has a top element (called the root) then the forest is said to be **rooted**.
- Finite forests are always rooted

Preorder forests with singletons roots

The cherry on top

- A **preorder forest with singleton roots** is a preorder P such that

$$P(x,y)\ \&\ \forall z(P(x,z) \Rightarrow P(z,x)\ \&\ P(y,z) \Rightarrow P(z,y)) \Rightarrow x = y$$

Now using these results about Boolean quantales, we have the following theorem:

Main Theorem

- Atomic Boolean commutative idempotent unital quantales are definitionally equivalent to preorder forests with singleton roots.
- In the **finite case** these algebras are Boolean cdi-semirings, hence all finite Boolean cdi-semirings can be constructed by enumerating preorder forests with singleton roots.

Finite forests

Note that every finite forest is a preorder forest with singleton roots.

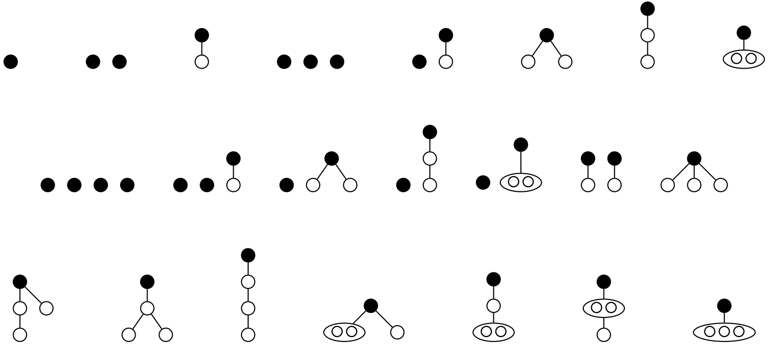


Figure: Preorder forests with singleton roots of size ≤ 4

- If a preorder forest has no bubbles then it corresponds to a partial-order, else it is a pre-order.

What we saw

Summary

- We showed that there are seven cdi-semirings with a $\vee$-semilattice of height less than or equal to 2.
- We constructed all cdi-semirings for which their multiplicative semilattice is a chain with $n+1$ elements, and we showed that up to isomorphism the number of such algebras is the n^{th} Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.
- The cdi-semirings that have a Boolean $\vee$-semilattice of size 2^n are determined by preorder forests with singleton roots.

Generalization: From Boolean to distributive quantales

Now we shall replace the set of atoms by a set of completely join-irreducible elements, and introduce the notion of PQ-frames in order to extend the Main Theorem and build even more cdi-semirings.

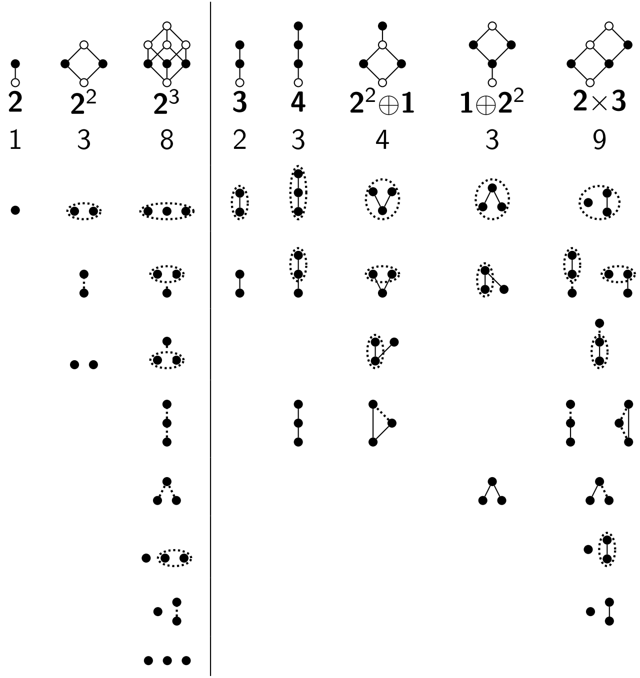
Completely join-irreducible

An element x is **completely join-irreducible** if $x = \bigvee S \implies x \in S$. Let $J(A)$ denote the set of completely join-irreducibles of A.

PQ-frame

$(W, \leq, P, Q)$ is a **PQ-frame** if

1. $(W, \leq)$ is a poset.
2. $P(x, y)$ & $x \leq u$ & $x \not\leq v$ & $y \leq v \implies P(u, v)$
3. $Q(x, y)$ & $x \leq u$ & $x \not\leq v$ & $y \leq v \implies Q(u, v)$
4. $x \leq y \implies P(x, y)$ & $Q(x, y)$


2
2^2
2^3
3
4
2^2⊕1
1⊕2^2
2×3
1
3
8
2
3
4
3
9

Thank you

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