Expressive Power and Succinctness of the Positive Calculus of Relations

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Introduction: CoR and FO^3

Our starting point is the following equivalence.

Thm (Tarski and Givant 1987). The following two have the same expressive power (in terms of binary relations):

- CoR (*Tarski's calculus of relations*),
- FO^3 (*three variable first-order logic*).

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Example:

- $a^{\smile} \equiv [a(y,x)]_{x,y}$

(converse)

$R^{\smile} := \{\langle w, v\rangle \mid \langle v, w\rangle \in R\}$

(composition)

$R \cdot S := \{\langle w, w'\rangle \mid \exists v. \langle w, v\rangle \in R \wedge \langle v, w'\rangle \in S\}$

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- $a \cdot b \equiv [\exists z. a(x, z) \land b(z, y)]_{x,y}$
- $a \cdot (a \cdot (a \cdot a)) \equiv [\exists z. a(x, z) \land \exists x. a(z, x) \land \exists z. a(x, z) \land a(z, y)]_{x,y}$

Cor. There exists a linear-size translation from CoR to FO^3.

universal relation | empty relation | identity relation

$$\mathrm{ST}_{x,y}(\top) := \mathrm{tt} \quad \mathrm{ST}_{x,y}(\mathbf{0}) := \mathrm{ff} \quad \mathrm{ST}_{x,y}(a) := a(x,y) \quad \mathrm{ST}_{x,y}(\mathbf{1}) := x = y$$
$$\mathrm{ST}_{x,y}(t^{-}) := \neg \mathrm{ST}_{x,y}(t) \quad \mathrm{ST}_{x_1,x_2}(t^{\pi}) := \mathrm{ST}_{x_{\pi(1)},x_{\pi(2)}}(t)$$
$$\mathrm{ST}_{x,y}(t \cup s) := \mathrm{ST}_{x,y}(t) \vee \mathrm{ST}_{x,y}(s) \quad \mathrm{ST}_{x,y}(t \cap s) := \mathrm{ST}_{x,y}(t) \wedge \mathrm{ST}_{x,y}(s)$$
$$\mathrm{ST}_{x,y}(t \cdot s) := \exists z. \mathrm{ST}_{x,z}(t) \wedge \mathrm{ST}_{z,y}(s) \quad \mathrm{ST}_{x,y}(t \dagger s) := \forall z. \mathrm{ST}_{x,z}(t) \vee \mathrm{ST}_{z,y}(s)$$

the dual of $\cdot$

The standard translation ($t \equiv [\mathrm{ST}_{x,y}(t)]_{x,y}$)

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The standard translation ($t \equiv [\mathrm{ST}_{x,y}(t)]_{x,y}$)

$\pi\colon [2] \to [2]$ denotes a *projection*: $R^{\pi} := \{\langle v_1, v_2\rangle \mid \langle v_{\pi(1)}, v_{\pi(2)}\rangle \in R\}$.

$$t^{\{1,2 \mapsto 1,2\}} := t \qquad t^{\{1,2 \mapsto 2,2\}} := \top \cdot (t \cap \mathbf{1})$$
$$t^{\{1,2 \mapsto 1,1\}} := (t \cap \mathbf{1}) \cdot \top \qquad t^{\{1,2 \mapsto 2,1\}} := t^{\smile}$$

How about *positive* fragments of CoR and FO^3?

(the positivity is also interesting in the view of the decidability.)

- (The equivalence of) Kleene allegory terms $(\bullet^{\smile}, 1, \cap, \cdot, 0, \cup, \bullet^{+})$ is decidable (Nakamura 2017).
- However, CoR is undecidable (Tarski and Givant 1987) (even for the just one binary relation case without the identity relation (Nakamura 2019)).

How about *positive* fragments of CoR and FO^3?

- They cannot be directly shown by existing translations (Tarski and Givant 1987; Maddux 2006; Givant 2017) from FO^3 into CoR.

PCoR(TC) (Pous 2018)
$(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup, \bullet^{+})$
lin-size →
← ?
EP^3(v-MTC)
$(=, \mathrm{tt}, \wedge, \exists, \mathrm{ff}, \vee, \text{v-MTC})$

PCoR
$(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup)$
lin-size →
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PPCoR (Pous and Vignudelli 2018)
$(\bullet^{\pi}, 1, \top, \cap, \cdot)$
lin-size →
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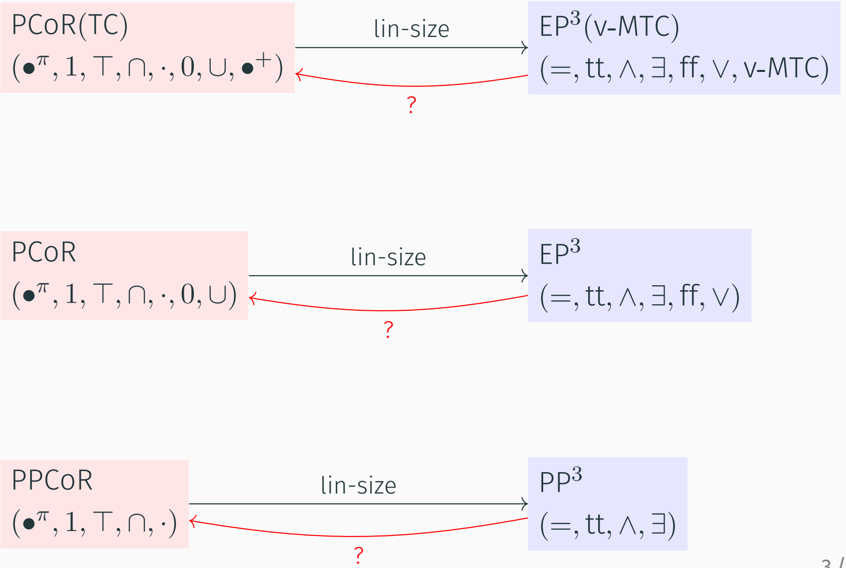
Outline

- ~~Introduction~~
- Contribution
- (1) A new translation from FO^3 to CoR
- (2) On succinctness
- Conclusion

Contribution



Contribution (1): Generalization for Positive Cases

PCoR(TC) $(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup, \bullet^{+})$ → lin-size → EP^3(v-MTC) $(=, \mathrm{tt}, \wedge, \exists, \mathrm{ff}, \vee, \text{v-MTC})$

EP^3(v-MTC) → **(1) exp-size [N '20]** → PCoR(TC)

PCoR $(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup)$ → lin-size → EP^3 $(=, \mathrm{tt}, \wedge, \exists, \mathrm{ff}, \vee)$

EP^3 → **(1) exp-size [N '20]** → PCoR

PPCoR $(\bullet^{\pi}, 1, \top, \cap, \cdot)$ → lin-size → PP^3 $(=, \mathrm{tt}, \wedge, \exists)$

PP^3 → **(1) lin-size [N '20]** → PPCoR

Contribution (2): On Succinctness

PCoR(TC) $(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup, \bullet^{+})$ —lin-size→ **EP^3(v-MTC)** $(=, \mathrm{tt}, \wedge, \exists, \mathrm{ff}, \vee, \text{v-MTC})$

(1) exp-size [N '20] (←)

(2) there doesn't exist any **poly-size** translation [N '20]

PCoR $(\bullet^{\pi}, 1, \top, \cap, \cdot, 0, \cup)$ —lin-size→ **EP^3** $(=, \mathrm{tt}, \wedge, \exists, \mathrm{ff}, \vee)$

(1) exp-size [N '20] (←)

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PPCoR $(\bullet^{\pi}, 1, \top, \cap, \cdot)$ ←lin-size→ **PP^3** $(=, \mathrm{tt}, \wedge, \exists)$

(1) lin-size [N '20]

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From FO^3 to CoR: Syntax and Semantics

Syntax:

$$t, s \in \mathrm{Term}^{\mathsf{CoR}} ::= t^{\pi} \mid a \mid \mathbf{1}$$
$$\mid \top \mid t \cap s \mid t \cdot s \mid \mathbf{0} \mid t \cup s \mid t \dagger s \mid t^{-}$$

$$\varphi, \psi \in \mathrm{Fml}^{\mathsf{FO}} ::= a(x, y) \mid x = y$$
$$\mid \mathrm{tt} \mid \varphi \wedge \psi \mid \exists x.\varphi \mid \mathrm{ff} \mid \varphi \vee \psi \mid \forall x.\varphi \mid \neg\varphi$$

$$t, s \in \mathrm{Term}^{\mathsf{FO}} ::= [\varphi]_{x,y} \quad \text{where } x \neq y \text{ and } \mathbf{FV}(\varphi) \subseteq \{x, y\}$$

Semantics:

$[\![t]\!]_M$: the binary relation of t on a structure M.

- Here, $[\![[\varphi]_{x,y}]\!]_M := \{\langle v, w\rangle \mid \{x \mapsto v, y \mapsto w\} \models_M \varphi\}$.
  - $I \models_M \varphi$ is the satisfaction relation (defined as usual).

We write $t \equiv s$ when $[\![t]\!]_M = [\![s]\!]_M$ for every M.

From FO^3 to CoR: Outline

FO^3

lin. ↓

1. Translate into *negation normal form*.

exp. ↓

2. For each sub-formula $\exists z.\psi$ (resp. $\forall z.\psi$), substitute ψ with an equivalent formula $\bigwedge_i \rho_i$ (resp. $\bigvee_i \rho_i$), where each ρ_i has at most *two* free variables.

lin. ↓

3. Push $\exists/\forall$ deeper into the formula as much as possible.

lin. ↓

4. Translate into a CoR term by simple structural induction.

CoR

From FO^3 to CoR: (1)

1. Translate into *negation normal form*.

Apply the De Morgan's law and the double negation elimination law repeatedly as much as possible:

$$\neg(\varphi \land \psi) \rightsquigarrow (\neg\varphi) \lor (\neg\psi) \qquad \neg\exists x.\varphi \rightsquigarrow \forall x.\neg\varphi$$

$$\neg(\varphi \lor \psi) \rightsquigarrow (\neg\varphi) \land (\neg\psi) \qquad \neg\forall x.\varphi \rightsquigarrow \exists x.\neg\varphi$$

$$\neg\neg\varphi \rightsquigarrow \varphi$$

Then, $\neg$ applies only to atomic formulas.

From FO^3 to CoR: (2)

2. For each sub-formula $\exists z.\psi$ (resp. $\forall z.\psi$), substitute ψ with an equivalent formula $\bigwedge_i \rho_i$ (resp. $\bigvee_i \rho_i$), where each ρ_i has at most *two* free variables.

Example:

$$\exists z.\ \underbrace{(a(x,z) \lor b(z,y))}_{✗\{x,y,z\}} \land \underbrace{c(y,z)}_{✓\{y,z\}}$$

2 For each sub-formula $\exists z.\psi$ (resp. $\forall z.\psi$), substitute ψ with an equivalent formula $\bigwedge_i \rho_i$ (resp. $\bigvee_i \rho_i$), where each ρ_i has at most *two* free variables.

Example:

$$\exists z.\, \underbrace{(a(x,z) \vee b(z,y))}_{✗\{x,y,z\}} \wedge \underbrace{c(y,z)}_{✓\{y,z\}}$$

$$\rightsquigarrow \exists z.\, \underbrace{(a(x,z) \wedge c(y,z)) \vee (b(z,y) \wedge c(y,z))}_{✗\{x,y,z\}} \qquad \text{(distribute } \wedge\text{-}\vee)$$

$$\rightsquigarrow (\exists z.\, \underbrace{a(x,z)}_{✓\{x,z\}} \wedge \underbrace{c(y,z)}_{✓\{y,z\}}) \vee (\exists z.\, \underbrace{b(z,y)}_{✓\{z,y\}} \wedge \underbrace{c(y,z)}_{✓\{y,z\}}) \qquad \text{(distribute } \exists\text{-}\vee)$$

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substitute ψ with an equivalent formula $\bigwedge_i \rho_i$ (resp. $\bigvee_i \rho_i$),
where each ρ_i has at most *two* free variables.

Example (for $\forall$):

$$\forall z.\, \underbrace{(a(x,z) \wedge b(z,y))}_{\text{✗}\{x,y,z\}} \vee \underbrace{c(y,z)}_{\text{✓}\{y,z\}}$$

$$\rightsquigarrow \forall z.\, \underbrace{(a(x,z) \vee c(y,z)) \wedge (b(z,y) \vee c(y,z))}_{\text{✗}\{x,y,z\}} \qquad \text{(distribute } \vee\text{-}\wedge\text{)}$$

$$\rightsquigarrow (\forall z.\, \underbrace{a(x,z)}_{\text{✓}\{x,z\}} \vee \underbrace{c(y,z)}_{\text{✓}\{y,z\}}) \wedge (\forall z.\, \underbrace{b(z,y)}_{\text{✓}\{z,y\}} \vee \underbrace{c(y,z)}_{\text{✓}\{y,z\}}) \qquad \text{(distribute } \forall\text{-}\wedge\text{)}$$

From FO^3 to CoR: (2)

2. For each sub-formula $\exists z.\psi$ (resp. $\forall z.\psi$), substitute ψ with an equivalent formula $\bigwedge_i \rho_i$ (resp. $\bigvee_i \rho_i$), where each ρ_i has at most *two* free variables.

- $$\mathsf{T}_\bullet(\varphi) := \begin{cases} \varphi & (\bullet = -) \\ \{\{\varphi\}\} & (\bullet = \exists, \forall) \end{cases} \text{ if } \varphi \text{ is an atomic or negated atomic formula.}$$
- $$\mathsf{T}_\bullet(\exists z.\varphi) := \begin{cases} \bigvee_{i \in [n]} \exists z. \bigwedge \Phi_i & (\bullet = -) \\ \{\{\bigvee_{i \in [n]} \exists z. \bigwedge \Phi_i\}\} & (\bullet = \exists, \forall) \end{cases}, \text{ where } \mathsf{T}_\exists(\varphi) = \{\Phi_i \mid i \in [n]\}.$$
- $$\mathsf{T}_\bullet(\forall z.\varphi) := \begin{cases} \bigwedge_{i \in [n]} \forall z. \bigvee \Phi_i & (\bullet = -) \\ \{\{\bigwedge_{i \in [n]} \forall z. \bigvee \Phi_i\}\} & (\bullet = \exists, \forall) \end{cases}, \text{ where } \mathsf{T}_\forall(\varphi) = \{\Phi_i \mid i \in [n]\}.$$
- $$\mathsf{T}_\bullet(\psi_1 \wedge \psi_2) := \begin{cases} \mathsf{T}_-(\psi_1) \wedge \mathsf{T}_-(\psi_2) & (\bullet = -) \\ \mathsf{T}_\forall(\psi_1) \cup \mathsf{T}_\forall(\psi_2) & (\bullet = \forall) \\ \{\Psi_1 \cup \Psi_2 \mid \Psi_1 \in \mathsf{T}_\exists(\psi_1), \Psi_2 \in \mathsf{T}_\exists(\psi_2)\} & (\bullet = \exists) \end{cases}$$
- $$\mathsf{T}_\bullet(\psi_1 \vee \psi_2) := \begin{cases} \mathsf{T}_-(\psi_1) \vee \mathsf{T}_-(\psi_2) & (\bullet = -) \\ \mathsf{T}_\exists(\psi_1) \cup \mathsf{T}_\exists(\psi_2) & (\bullet = \exists) \\ \{\Psi_1 \cup \Psi_2 \mid \Psi_1 \in \mathsf{T}_\forall(\psi_1), \Psi_2 \in \mathsf{T}_\forall(\psi_2)\} & (\bullet = \forall) \end{cases}$$

From FO^3 to CoR: (3)

3 Push $\exists/\forall$ deeper into the formula as much as possible.

Example:

$$\exists z.\varphi_{\sigma^{-1}(1)} \wedge \cdots \wedge \varphi_{\sigma^{-1}(n)}$$

$$\rightsquigarrow \exists z.\underbrace{\varphi_1 \wedge \cdots \wedge \varphi_{n'}}_{\{x,y\}} \wedge \underbrace{\varphi_{n'+1} \wedge \cdots \wedge \varphi_{n''}}_{\{x,z\}} \wedge \underbrace{\varphi_{n''+1} \wedge \cdots \wedge \varphi_{n}}_{\{z,y\}}$$

(permutate $\varphi_1, \ldots, \varphi_n$)

$$\rightsquigarrow \underbrace{\varphi_1 \wedge \cdots \wedge \varphi_{n'}}_{\{x,y\}} \wedge \exists z.\underbrace{\varphi_{n'+1} \wedge \cdots \wedge \varphi_{n''}}_{\{x,z\}} \wedge \underbrace{\varphi_{n''+1} \wedge \cdots \wedge \varphi_{n}}_{\{z,y\}}$$

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Then, the formula is in the formula set defined by: (where $w, w' \in \{x, y\}$)

$$\varphi^{\{x,y\}}, \psi^{\{x,y\}} ::= a(w,w') \mid \neg a(w,w') \mid w = w' \mid \neg w = w' \mid \mathrm{tt} \mid \mathrm{ff} \mid \varphi^{\{x,y\}} \vee \psi^{\{x,y\}}$$

$$\mid \varphi^{\{x,y\}} \wedge \psi^{\{x,y\}} \mid \underbrace{\exists z.\varphi^{\{x,z\}} \wedge \psi^{\{z,y\}}}_{(\cdot)} \mid \underbrace{\forall z.\varphi^{\{x,z\}} \vee \psi^{\{z,y\}}}_{(\dagger)}.$$

From FO^3 to CoR: (4)

4. Translate into a CoR term by simple structural induction.

$$\varphi^{\{x,y\}}, \psi^{\{x,y\}} ::= a(w, w') \mid \neg a(w, w') \mid w = w' \mid \neg w = w' \mid \mathrm{tt} \mid \mathrm{ff} \mid \varphi^{\{x,y\}} \vee \psi^{\{x,y\}} \\ \mid \varphi^{\{x,y\}} \wedge \psi^{\{x,y\}} \mid \exists z. \varphi^{\{x,z\}} \wedge \psi^{\{z,y\}} \mid \forall z. \varphi^{\{x,z\}} \vee \psi^{\{z,y\}}.$$

$$\mathsf{T}([a(x_i, x_j)]_{x_1,x_2}) := a^\pi \quad \mathsf{T}([\neg a(x_i, x_j)]_{x_1,x_2}) := (a^-)^\pi \quad \text{where } \pi = \{1 \mapsto i, 2 \mapsto j\}$$

$$\mathsf{T}([x_i = x_j]_{x_1,x_2}) := \mathbf{1}^\pi \quad \mathsf{T}([\neg x_i = x_j]_{x_1,x_2}) := (\mathbf{1}^-)^\pi$$

$$\mathsf{T}([\mathrm{tt}]_{x,y}) := \top \quad \mathsf{T}([\varphi^{\{x,y\}} \wedge \psi^{\{x,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,y\}}]_{x,y}) \cap \mathsf{T}([\psi^{\{x,y\}}]_{x,y})$$

$$\mathsf{T}([\mathrm{ff}]_{x,y}) := \mathbf{0} \quad \mathsf{T}([\varphi^{\{x,y\}} \vee \psi^{\{x,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,y\}}]_{x,y}) \cup \mathsf{T}([\psi^{\{x,y\}}]_{x,y})$$

$$\mathsf{T}([\exists z. \varphi^{\{x,z\}} \wedge \psi^{\{z,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,z\}}]_{x,z}) \cdot \mathsf{T}([\psi^{\{z,y\}}]_{z,y})$$

$$\mathsf{T}([\forall z. \varphi^{\{x,z\}} \vee \psi^{\{z,y\}}]_{x,y}) := \mathsf{T}([\varphi^{\{x,z\}}]_{x,z}) \dagger \mathsf{T}([\psi^{\{z,y\}}]_{z,y})$$

From FO^3 to CoR: Result

PCoR(TC) $(\bullet^\pi, 1, \top, \cap, \cdot, 0, \cup, \bullet^+)$ → lin-size → EP^3(v-MTC) $(=, tt, \wedge, \exists, ff, \vee, \text{v-MTC})$

EP^3(v-MTC) → **exp-size [N '20]** → PCoR(TC)

(use the translations (2,3,4) with a bit extension for TC)

PCoR $(\bullet^\pi, 1, \top, \cap, \cdot, 0, \cup)$ → lin-size → EP^3 $(=, tt, \wedge, \exists, ff, \vee)$

EP^3 → **exp-size [N '20]** → PCoR

(use the translations (2,3,4))

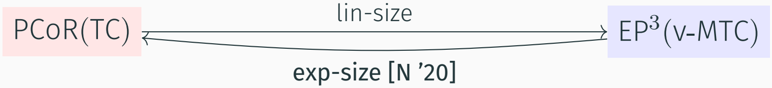
PPCoR $(\bullet^\pi, 1, \top, \cap, \cdot)$ ↔ lin-size / **lin-size [N '20]** ↔ PP^3 $(=, tt, \wedge, \exists)$

(use the translations (3,4))

Corollary (1): $\mathsf{PCoR(TC)} \longleftrightarrow \mathsf{EP}^3(\mathsf{v\text{-}MTC})$

$$t, s \in \mathsf{Term}^{\mathsf{PCoR(TC)}} ::= \cdots \mid t^{+}$$

$$\varphi, \psi \in \mathsf{Fml}^{\mathsf{EP(v\text{-}MTC)}} ::= \cdots \mid [\mathsf{TC}_{z,u}(\varphi)](x, y) \quad \text{where } \mathbf{FV}(\varphi) \subseteq \{z, u\}$$

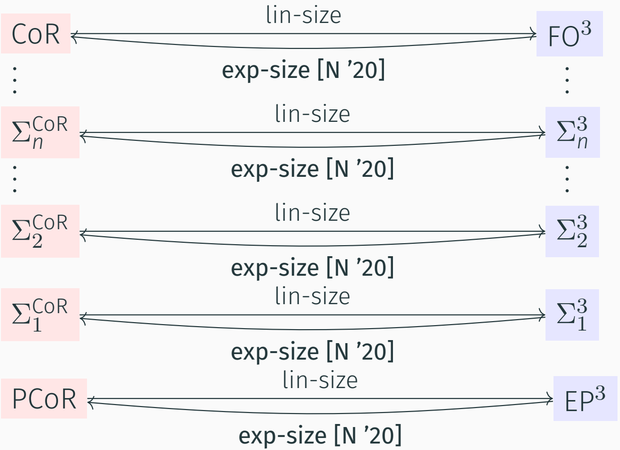


- $\mathsf{PCoR(TC)} \rightsquigarrow \mathsf{EP}^3(\mathsf{v\text{-}MTC})$:
  By the standard translation with
  $\mathrm{ST}_{x,y}(t^{+}) := [\mathsf{TC}_{x,y}(\mathrm{ST}_{x,y}(t))](x, y)$
- $\mathsf{EP}^3(\mathsf{v\text{-}MTC}) \rightsquigarrow \mathsf{PCoR(TC)}$:
  By (2-4) with
  $\mathrm{T}([[\mathsf{TC}_{z,u}(\varphi)](x_i, x_j)]_{x_1,x_2}) := (\mathrm{T}([\varphi]_{z,u})^{+})^{\pi}$ in (4)
  where $\pi = \{1 \mapsto i, 2 \mapsto j\}$.

Corollary (2): the dot-dagger alternation hierarchy

Σ_n: the quantifier ($\exists/\forall$) alternation hierarchy.

- Σ_n^3: analogy of Σ_n for FO^3.
- Σ_n^{CoR}: analogy of Σ_n for CoR ($\exists$ / $\forall$ $\rightsquigarrow$ $\cdot$ / $\dagger$).



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On succinctness of PCoR: (1)

Thm. There is no $2^{o(n)}$-size translation from EP^3 terms to equivalent PCoR terms.

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Thm. There is no $2^{o(n)}$-size translation from EP^3 terms to equivalent PCoR terms.

Observation: Let $\mathrm{Part}(X) := \{\langle I, J\rangle \mid I \cup J = X, I \cap J = \emptyset\}$. Then,

$$\left[\exists z. \left(\bigwedge_{i \in [n]} \mathsf{a}_i(x, z) \vee \mathsf{b}_i(z, y)\right)\right]_{x,y}$$

$$\equiv \left[\bigvee_{\langle I,J\rangle \in \mathrm{Part}([n])} \exists z. (\bigwedge_{i \in I} \mathsf{a}_i(x, z)) \wedge (\bigwedge_{j \in J} \mathsf{b}_j(z, y))\right]_{x,y} \qquad \text{(step 2)}$$

$$\equiv \left(\bigcup_{\langle I,J\rangle \in \mathrm{Part}([n])} (\bigcap_{i \in I} \mathsf{a}_i) \cdot (\bigcap_{j \in J} \mathsf{b}_j)\right) \qquad \text{(step 4).}$$

But, $\#(\mathrm{Part}([n])) = 2^n$ is exponential in n.
Hence, this is not a $2^{o(n)}$-size translation! 😟

Thm. There is no $2^{o(n)}$-size translation from EP^3 terms to equivalent PCoR terms.

$\Uparrow$

Lem. There is no $2^{o(n)}$-size translation from $\{\mathfrak{t}_n \mid n \in \mathbb{N}_+\}$ to equivalent PCoR terms. Here, $\mathfrak{t}_n$ is

$$\left[\underbrace{(x = y)}_{\mathrm{d}(x,y) = 0} \vee \underbrace{\left(\bigvee_{i \in [0,n]} \mathsf{a}_i(x,y) \vee \mathsf{b}_i(x,y) \right)}_{\mathrm{d}(x,y) = 1} \vee \underbrace{\exists z.\ \left(\mathsf{a}_0(x,z) \wedge \mathsf{b}_0(z,y) \wedge \bigwedge_{i \in [n]} \mathsf{a}_i(x,z) \vee \mathsf{b}_i(z,y) \right)}_{\mathrm{d}(x,y) = 2} \right]_{x,y}.$$

On succinctness of PCoR: (3)

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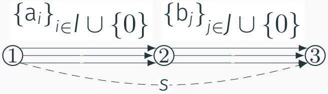
We use the following parameter $\mathrm{w}_n(s)$:

$$\#\left(\left\{ \langle I, J \rangle \in \mathrm{Part}([n]) \mid \right. \right.$$



$$\left. \left. \right\} \right) .$$

Note that $\mathrm{w}_n(\mathfrak{t}_n) = 2^n$.

On succinctness of PCoR: (3)

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$\Uparrow$

Lem. For every PCoR term s, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq 8\|s\|$.

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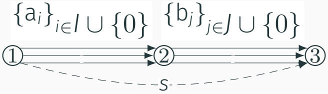
$$\# \left(\left\{ \langle I, J \rangle \in \mathrm{Part}([n]) \mid \right. \right.$$



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$$\left[\underbrace{(x = y)}_{\mathrm{d}(x,y)=0} \lor \underbrace{\left(\bigvee_{i \in [0,n]} \mathsf{a}_i(x,y) \lor \mathsf{b}_i(x,y) \right)}_{\mathrm{d}(x,y)=1} \lor \underbrace{\exists z. \left(\mathsf{a}_0(x,z) \land \mathsf{b}_0(z,y) \land \bigwedge_{i \in [n]} \mathsf{a}_i(x,z) \lor \mathsf{b}_i(z,y) \right)}_{\mathrm{d}(x,y)=2} \right]_{x,y} .$$

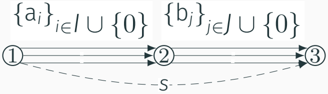
$\Uparrow$

Lem. For every PCoR term s, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq 8\|s\|$.

When $s \equiv \mathfrak{t}_n$, $2^n = \mathrm{w}_n(\mathfrak{t}_n) = \mathrm{w}_n(s) \leq 8\|s\|$.

So, $\|s\|$ is at least exponential in n.

We use

$$\#\left(\left\{ \langle I, J \rangle \in \mathrm{Part}([n]) \mid \right.\right.$$


$$\left.\left.\right\}\right) .$$

Note that $\mathrm{w}_n(\mathfrak{t}_n) = 2^n$.

On succinctness of PCoR: (4)

Lem. For every PCoR term s,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq 8\|s\|$.

PCoR term
$t, s ::= t^{\pi} \mid a \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s.$

$\Uparrow$

Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Projection normal form
$t, s ::= a \mid a^{\smile} \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s.$

On succinctness of PCoR: (4)

Lem. For every PCoR term s,

if $\models s \leq \mathfrak{t}_n$, th

PCoR term

$t, s ::=$ t^{π} $\mid a$

$\Uparrow$

$$t^{\{1,2 \mapsto 1,2\}} \leadsto t \qquad t^{\{1,2 \mapsto 2,2\}} \leadsto \top \cdot (t \cap \mathbf{1})$$

$$t^{\{1,2 \mapsto 1,1\}} \leadsto (t \cap \mathbf{1}) \cdot \top \qquad t^{\{1,2 \mapsto 2,1\}} \leadsto t^{\smile}$$

$$t^{\smile\smile} \leadsto t \quad \mathbf{1}^{\smile} \leadsto \mathbf{1} \quad \top^{\smile} \leadsto \top \quad \mathbf{0}^{\smile} \leadsto \mathbf{0}$$

$$(t \cup s)^{\smile} \leadsto t^{\smile} \cup s^{\smile} \quad (t \cap s)^{\smile} \leadsto t^{\smile} \cap s^{\smile} \quad (t \cdot s)^{\smile} \leadsto s^{\smile} \cdot t^{\smile}.$$

Lem. For every PCoR term s in projection normal form,

if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Projection normal form

$t, s ::= a \mid a^{\smile} \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s.$

Lem. For every PCoR term s,
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PCoR term
$t, s ::= t^{\pi} \mid a \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s.$

$\Uparrow$

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Projection normal form
$t, s ::= a \mid a^{\smile} \mid \mathbf{1} \mid \top \mid \mathbf{0} \mid t \cup s \mid t \cap s \mid t \cdot s.$

Proof. By induction on the structure of s.

Inductive cases.

Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. By induction on the structure of s.

Lem (For induction).

1. if $\models s_1 \cup s_2 \leq \mathfrak{t}_n$, $\models s_1 \leq \mathfrak{t}_n$ and $\models s_2 \leq \mathfrak{t}_n$.

2. if $\models s_1 \cap s_2 \leq \mathfrak{t}_n$, $\models s_1 \leq \mathfrak{t}_n$ or $\models s_2 \leq \mathfrak{t}_n$.

3. if $\models s_1 \cdot s_2 \leq \mathfrak{t}_n$ and $\not\models s_1 \cdot s_2 = \mathbf{0}$, $\models s_1 \leq \mathfrak{t}_n$ and $\models s_2 \leq \mathfrak{t}_n$.

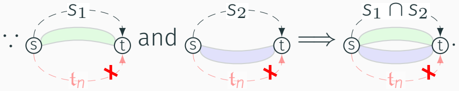
On succinctness of PCoR: (5)

Lem. For every PCoR term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

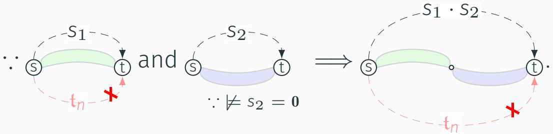
Proof. By induction on the structure of s.

Lem (For induction).

1. if $\models s_1 \cup s_2 \leq \mathfrak{t}_n$, $\models s_1 \leq \mathfrak{t}_n$ and $\models s_2 \leq \mathfrak{t}_n$.
   $\because$ Trivial.
2. if $\models s_1 \cap s_2 \leq \mathfrak{t}_n$, $\models s_1 \leq \mathfrak{t}_n$ or $\models s_2 \leq \mathfrak{t}_n$.



3. if $\models s_1 \cdot s_2 \leq \mathfrak{t}_n$ and $\not\models s_1 \cdot s_2 = \mathbf{0}$, $\models s_1 \leq \mathfrak{t}_n$ and $\models s_2 \leq \mathfrak{t}_n$.



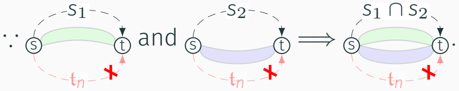
Lem. For every PCoR term s in projection normal form, if $\models s \le \mathfrak{t}_n$, then $\mathrm{w}_n(s) \le \|s\|$.

Proof. By induction on the structure of s.

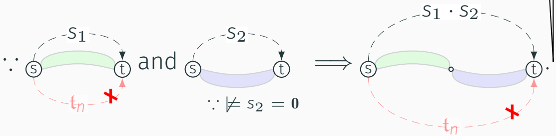
Lem (For induction).

(Actually, $\mathfrak{t}_n$ is defined for satisfying them.)

1. if $\models s_1 \cup s_2 \le \mathfrak{t}_n$, $\models s_1 \le \mathfrak{t}_n$ and $\models s_2 \le \mathfrak{t}_n$.
   $\because$ Trivial.
2. if $\models s_1 \cap s_2 \le \mathfrak{t}_n$, $\models s_1 \le \mathfrak{t}_n$ or $\models s_2 \le \mathfrak{t}_n$.



3. if $\models s_1 \cdot s_2 \le \mathfrak{t}_n$ and $\not\models s_1 \cdot s_2 = \mathbf{0}$, $\models s_1 \le \mathfrak{t}_n$ and $\models s_2 \le \mathfrak{t}_n$.

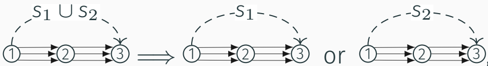


On succinctness of PCoR: (6)

Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. By induction on the structure of s.

Case $s = s_1 \cup s_2$:

By 

,

$$
\begin{aligned}
\mathrm{w}_n(s_1 \cup s_2) &\leq \mathrm{w}_n(s_1) + \mathrm{w}_n(s_2) \\
&\leq \|s_1\| + \|s_2\| && \text{(I.H.)} \\
&\leq \|s_1 \cup s_2\| && \square
\end{aligned}
$$

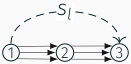
On succinctness of PCoR: (6)

Lem. For every PCoR term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. By induction on the structure of s.

Case $s = s_1 \cap s_2$: Let $l \in \{1, 2\}$ be such that $\models s_l \leq \mathfrak{t}_n$.

By 

$\Longrightarrow$ 

for each $l \in \{1, 2\}$,

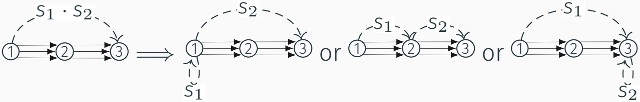
$$
\begin{aligned}
\mathrm{w}_n(s_1 \cap s_2) &\leq \mathrm{w}_n(s_l) \\
&\leq \|s_l\| && \text{(I.H.)} \\
&\leq \|s_1 \cap s_2\| && \square
\end{aligned}
$$

On succinctness of PCoR: (6)

Lem. For every PCoR term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. By induction on the structure of s.

Case $s = s_1 \cdot s_2$:

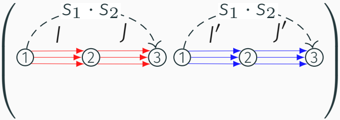
Observation: 

On succinctness of PCoR: (6)

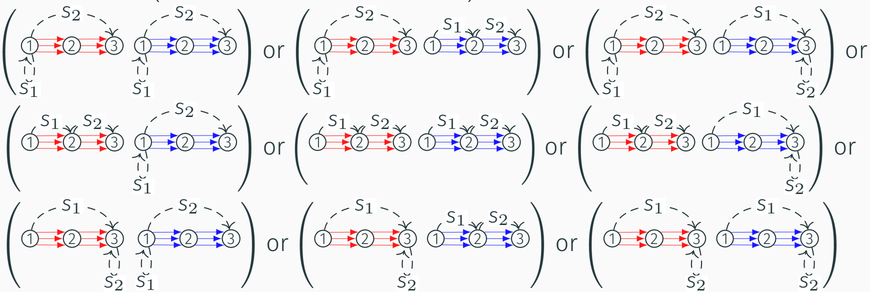
Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. By induction on the structure of s.

Case $s = s_1 \cdot s_2$: $\langle I, J \rangle, \langle I', J' \rangle \in \mathrm{Part}([n])$

Observation: 

$\Rightarrow$



Lem. For every PCoR term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

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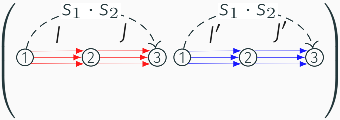
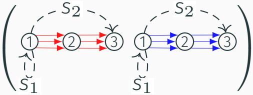
Observation: 

$\Rightarrow$



or or or

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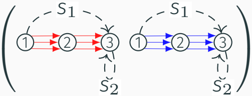
or or 

Lem. For every PCoR term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

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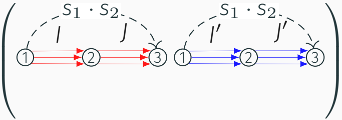
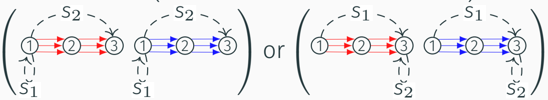
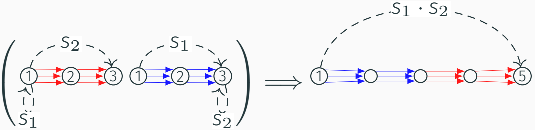
Observation:



$\Rightarrow$



$\because$



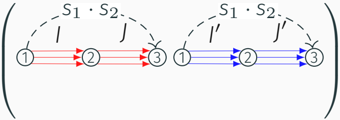
$\Longrightarrow$ This contradicts to $\models s_1 \cdot s_2 \leq \mathfrak{t}_n$ ($\because \mathrm{d}(1,5) > 2$).

On succinctness of PCoR: (6)

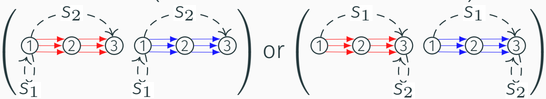
Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

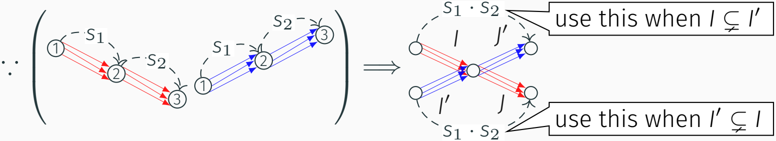
Proof. By induction on the structure of s.

Case $s = s_1 \cdot s_2$: $\langle I, J \rangle, \langle I', J' \rangle \in \mathrm{Part}([n])$

Observation: 

$\Rightarrow$



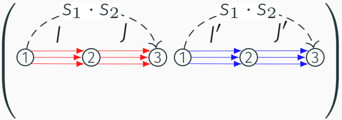


$\Longrightarrow$ This contradicts to $\models s_1 \cdot s_2 \leq \mathfrak{t}_n$.

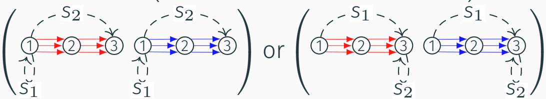
Lem. For every PCoR term s in projection normal form, if $\models s \le \mathfrak{t}_n$, then $\mathrm{w}_n(s) \le \|s\|$.

Proof. By induction on the structure of s.

Case $s = s_1 \cdot s_2$: $\langle I, J\rangle, \langle I', J'\rangle \in \mathrm{Part}([n])$

Observation: 

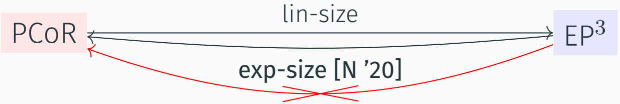
$\Rightarrow$



$\therefore$ When $\mathrm{w}_n(s_1 \cdot s_2) \ge 2$ (and $\not\models s_1 \cdot s_2 = \mathbf{0}$),

$$\begin{aligned}
\mathrm{w}_n(s_1 \cdot s_2) &\le \max(\mathrm{w}_n(s_2), \mathrm{w}_n(s_1)) && \text{(by using the above observation)}\\
&\le \max(\|s_2\|, \|s_1\|) && \text{(I.H.)}\\
&\le \|s_1 \cdot s_2\| && \square
\end{aligned}$$

Summary



$\Uparrow$

Lem. There is no $2^{o(n)}$-size translation from $\{\mathfrak{t}_n \mid n \in \mathbb{N}_+\}$ to equivalent PCoR terms.

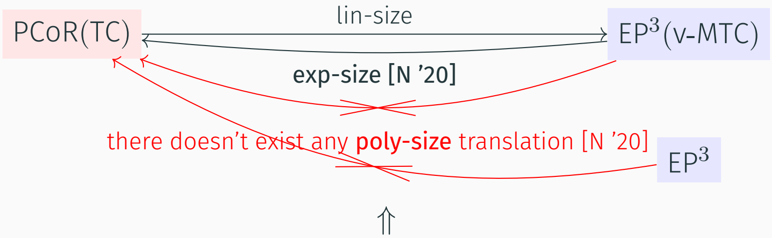
$\Uparrow$

Lem. For every PCoR term s,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq 8\|s\|$.

$\Uparrow$

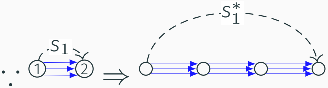
Lem. For every PCoR term s in projection normal form,
if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$. □

Corollary: on succinctness of PCoR(TC)



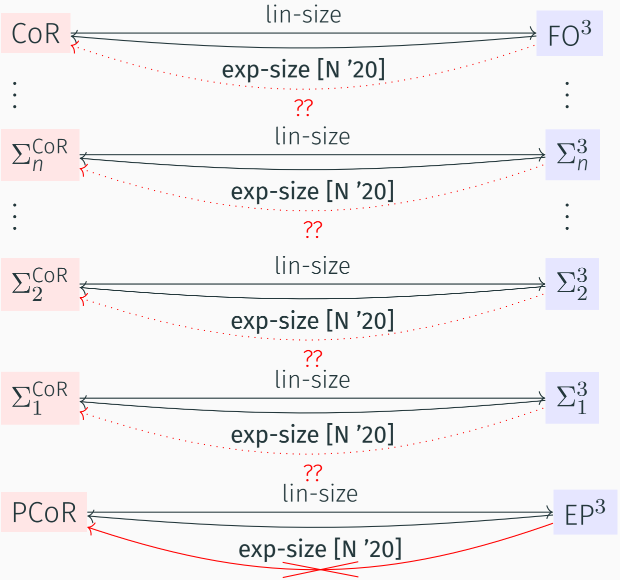
Lem. For every PCoR(TC) term s in projection normal form, if $\models s \leq \mathfrak{t}_n$, then $\mathrm{w}_n(s) \leq \|s\|$.

Proof. Case $s = s_1^*$: Then $\mathrm{w}_n(s_1^*) = 0$ ($\because \models s_1 \leq \mathbf{1}$).



$\Rightarrow$ This contradicts to $\models s_1^* \leq \mathfrak{t}_n$. $\square$

Future work: on succinctness in the dot-dagger hierarchy

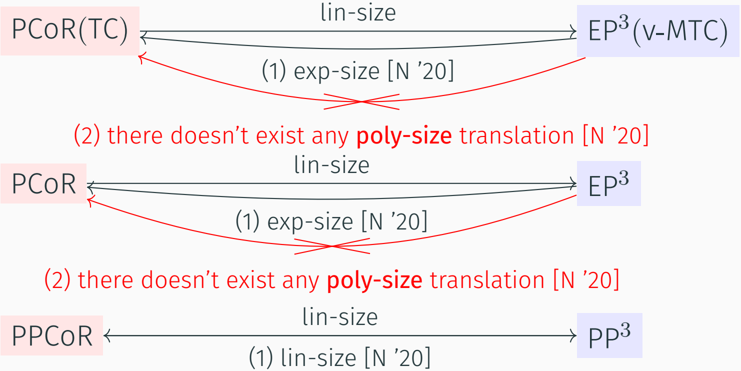


There doesn't exist any **poly-size** translation [N '20]

Outline

- ~~Introduction~~
- ~~Contribution~~
- ~~(1) A new translation from FO^3 to CoR~~
- ~~(2) On succinctness~~
- Conclusion

Conclusion: We have shown that:



Future Work:

- On the succinctness gap beyond positive cases.
  - CoR is exponentially less succinct than FO^3? (← formula-size game)
- To show some complexity gap between PCoR and EP^3.
- Some PCoR for EP^k (in terms of the above relationship).

Thank you for your attention!

References

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