A Hierarchy of Algebras for Boolean Subsets

Walter Guttmann and Bernhard Möller

RAMiCS 2020

Introduction

- Boolean algebra axiomatises logical operations such as

$$\vee, \wedge, \rightarrow, \neg, \mathsf{nand}, \mathsf{xor}, 0, 1$$

- there are different bases for this
- options
  - simple but many axioms (examples below)
  - few but complex axioms: e.g. Huntington, Sheffer

Plan

- ► axioms for Boolean algebras
- axioms for Boolean subsets
- stronger axioms for Boolean subsets

Simple but Many Axioms

- bounded semilattices

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z \qquad x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z$$
$$x \sqcup y = y \sqcup x \qquad x \sqcap y = y \sqcap x$$
$$x \sqcup x = x \qquad x \sqcap x = x$$
$$x \sqcup \bot = x \qquad x \sqcap \top = x$$

- bounded distributive lattice

$$x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z) \qquad x \sqcup (x \sqcap y) = x$$
$$x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z) \qquad x \sqcap (x \sqcup y) = x$$

- complemented distributive lattice

$$x \sqcup \overline{x} = \top \qquad x \sqcap \overline{x} = \bot$$

Complex but Few Axioms

- Huntington:

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$$

$$x \sqcup y = y \sqcup x$$

$$x = \overline{\overline{x} \sqcup y} \sqcup \overline{\overline{x} \sqcup \overline{y}}$$

- Sheffer stroke (nand):

$$(x|((y|x)|x))|(y|(z|x)) = y$$

- many others

compromise

- Byrne:

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$$

$$x \sqcup y = y \sqcup x$$

$$x \sqcup \overline{y} = z \sqcup \overline{z} \;\Leftrightarrow\; x \sqcup y = x$$

- derived notions

$$\top \quad = \quad z \sqcup \overline{z}$$

$$y \sqsubseteq x \quad \Leftrightarrow \quad x \sqcup \overline{y} = \top$$

Equational Compromise (Current Paper)

- semilattice

$$
\begin{aligned}
x \sqcup (y \sqcup z) &= (x \sqcup y) \sqcup z \\
x \sqcup y &= y \sqcup x \\
x \sqcup x &= x
\end{aligned}
$$

- complement

$$
\begin{aligned}
\overline{\overline{x}} &= x \\
x \sqcup \overline{x} &= y \sqcup \overline{y} \\
x \sqcup \overline{x \sqcup y} &= x \sqcup \overline{y} \\
x \sqcup (\overline{x} \sqcap \overline{y}) &= x \sqcup \overline{y}
\end{aligned}
$$

Plan

- axioms for Boolean algebras
- ► axioms for Boolean subsets
- stronger axioms for Boolean subsets

Examples of Boolean Subsets

motivation: algebraic program semantics / program derivation

- state changes vs. states

actions/commands/ statements	tests/conditions/ propositions
Kleene algebra	Boolean algebra

- graphs

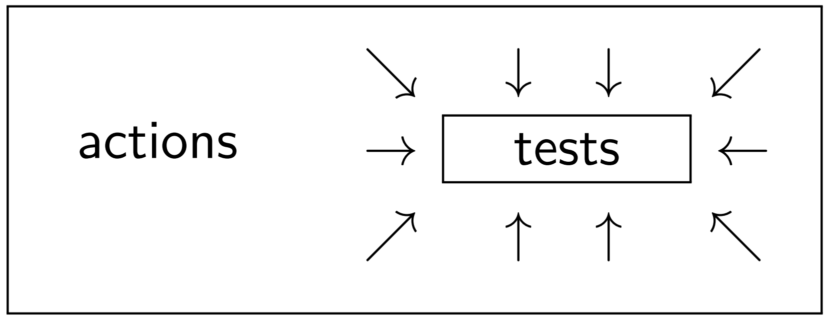
weighted graphs	relations/ unweighted graphs
Stone algebra	Boolean algebra

Kleene Algebra with Tests

- two sorts
- $(K, +, \cdot, ^*, 0, 1)$ Kleene algebra
- $(B, +, \cdot, \overline{}, 0, 1)$ Boolean algebra
- $B \subseteq K$
- problem: many ATPs do not know sorts
- approach: have subset of tests induced as the range of a special operation

Antidomain Semirings

- tests = range of (anti)domain operation



- axioms (a antidomain, $d = a^2$ domain)

$$a(x) \cdot x = 0$$

$$a(x) + d(x) = 1$$

$$d(x) \leq a(a(x) \cdot y)$$

- then the range of a is a Boolean algebra

Test Algebras

- avoid semiring/Kleene algebra structure on overall set as well as (anti)domain operation
- consider $(S, \cdot, {}')$ such that (Huntington)

$$x' \cdot (y' \cdot z') = (x' \cdot y') \cdot z'$$
$$x' \cdot y' = y' \cdot x'$$
$$x' = (x'' \cdot y')' \cdot (x'' \cdot y'')'$$
$$x' \cdot y' = (x' \cdot y')''$$

- range of $'$ is a Boolean algebra
- any other axiomatisation instead of Huntington's can be used

Abstraction: B_1 Algebras

we exhibit a common ground for (anti)domain and graph structures

- overall set a semilattice (holds, e.g., in all IL-semirings)

$$x \sqcup (y \sqcup z) = (x \sqcup y) \sqcup z$$

$$x \sqcup y = y \sqcup x$$

$$x \sqcup x = x$$

- subset (range of $\overline{}$) a Boolean algebra (our equational version)

$$\overline{\overline{\overline{x}}} = \overline{x}$$

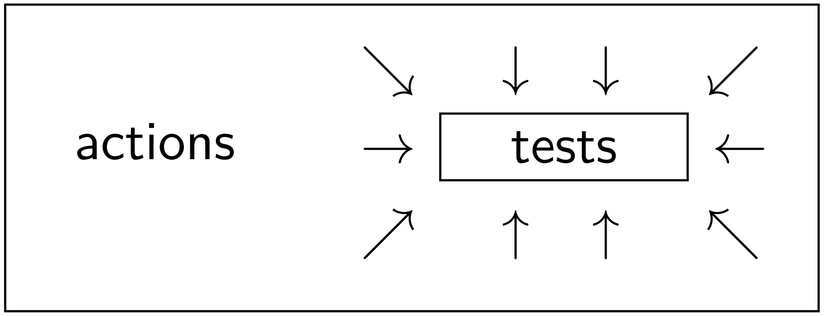
$$\overline{x \sqcup \overline{x}} = \overline{y \sqcup \overline{y}}$$

$$\overline{x} \sqcup \overline{\overline{x} \sqcup y} = \overline{x} \sqcup \overline{y}$$

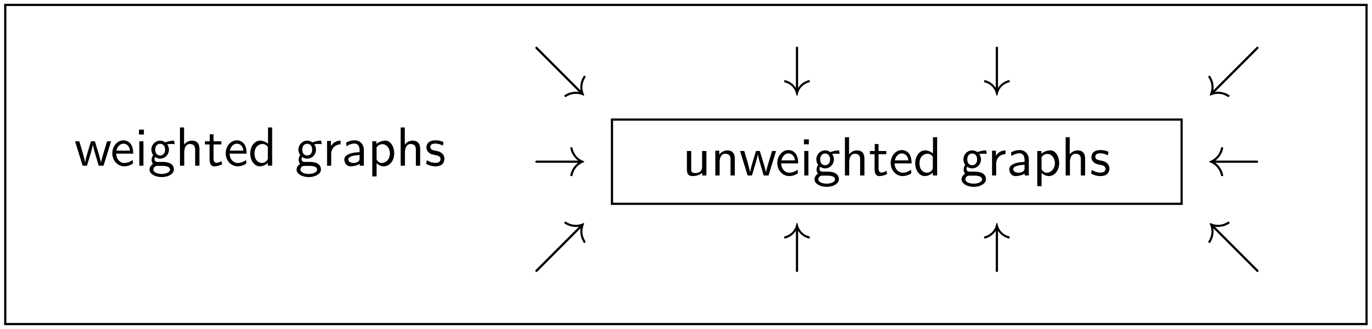
- a bit stronger than just subset Boolean algebra

Instances

- tests are the range of $a/'$



- unweighted graphs are the range of pseudocomplement $^{-}$



Plan

- axioms for Boolean algebras
- axioms for Boolean subsets
- ► stronger axioms for Boolean subsets

Boolean Subsets with Additional Structure

- examples discussed above (tests, weighted graphs) satisfy further common properties
- in these models, the unary operation can be a complement, a pseudocomplement or the antidomain operation
- for simplicity, we mostly call $\overline{}$ the *complement*

- we build a hierarchy of algebras with stronger and stronger assumptions on meet and complement

B_1: semilattice + “basic” complement

B_2: De Morgan

B_3: pre-IL-semiring with weak antidomain

B_4: IL-semiring with antidomain

- this provides a more structured path from test algebras to antidomain algebras

B_2 Algebras: De Morgan

axioms: semilattice plus

$$\begin{aligned} x \sqcup \overline{y \sqcup \overline{y}} &= x \\ \overline{x \sqcup y} &= \overline{\overline{\overline{x}} \sqcup \overline{\overline{y}}} \\ \overline{x} \sqcup \overline{\overline{x} \sqcup y} &= \overline{x} \sqcup \overline{y} \end{aligned}$$

Theorem every B_2 algebra is a B_1 algebra

B_3 Algebras: Pre-IL-Semirings with Weak Antidomain

for B_3 we supply a translation table for the operations and relations

B_3-algebra	antidom. mod.	B_3-algebra	antidom. mod.
$\sqcup$	$+$	$\bot$	0
$\sqcap$	$\cdot$	$\top$	1
$^-$	a	$\sqsubseteq$	$\leq$
$^=$	d	$\sqsubset$	$<$

the additional B_3 axioms:

$$x \sqcap (y \sqcap z) = (x \sqcap y) \sqcap z \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

$$(x \sqcup y) \sqcap z = (x \sqcap z) \sqcup (y \sqcap z) \qquad (x + y) \cdot z = (x \cdot z) + (y \cdot z)$$

$$\overline{x} \sqcap x = \bot \qquad a(x) \cdot x = 0$$

$$\top \sqcap x = x \qquad 1 \cdot x = x$$

$$\overline{x \sqcap \overline{\overline{y}}} = \overline{x \sqcap y} \qquad a(x \cdot d(y)) = a(x \cdot y)$$

B_4 Algebras: IL-Semirings with Antidomain

the additional B_4 axioms

$$x \sqcap \top = x \qquad\qquad x \cdot 1 = x$$

$$x \sqsubseteq y \Rightarrow z \sqcap x \sqsubseteq z \sqcap y \qquad\qquad x \leq y \Rightarrow z \cdot x \leq z \cdot y$$

Theorem B_4 algebras are equivalent to IL-semirings with antidomain

B_5 Algebras: Meet and Pseudocomplement

- B_1: semilattice + "basic" complement
  - B_2: De Morgan
    - B_3: pre-IL-semiring with weak antidomain
      - B_4: IL-semiring with antidomain
      - B_5: meet and pseudocomplement

the additional B_5 axioms

$$x \sqcap y = y \sqcap x \qquad x \sqcap (x \sqcup y) = x$$

Theorem

- every B_5 algebra is a B_4 algebra
- B_5 algebras are equivalent to Stone algebras
- hence every Stone algebra is an antidomain IL-semiring

Concluding Remarks

achievements

- hierarchy of axiom systems as a common basis for inducing a Boolean subalgebra in a larger overall algebra as the range of a complement-like operation
- this has shed new light on the interconnections between several such approaches
- axioms simple and perspicuous when translated into formulas of the respective theories
- only equational axioms, hence well suited to mechanical support
- All results proved with Isabelle and Prover9
  `https://www.isa-afp.org/entries/Subset_Boolean_Algebras.html`

outlook

- In several cases, Prover9 was able to find a proof where the tools called by Sledgehammer failed
- since Prover9 is not integrated with Sledgehammer, we wrote a program to transform Prover9 output into Isabelle/HOL proofs
- this currently works for a limited range of proofs but could form the basis of an integration into Sledgehammer
- such an extension would be beneficial because Prover9 performs well for algebraic applications